Chapter 30 of the booklet ***General Theory of Relativity*** by P.A.M. Dirac
PRINCETON PAPERBACKS ISBN 0-691-01146-X

Below a copy of chapter 30 is given with links to other chapters. If one wants to understand the background of GR please read this booklet first. In chapter 30 Einstein's Comprehensive Action Principle is explained. The CAP implies that curvature of 4D-spacetime must always be included in any mathematical analysis of physics.

30. The comprehensive action principle

The method of Section 29 can be generalized to apply to the gravitational field interacting with any other fields, which are also interacting with one another. There is a comprehensive action principle,

$$\delta(I_g + I') = 0, \qquad (30.1)$$

where I_g is the gravitational action that we had before and I' is the action of all the other fields and consists of a sum of terms, one for each field. It is a great advantage of using an action principle that it is so easy to obtain the correct equations for any fields in interaction. One merely has to obtain the action for each of the fields one is interested in and add them all together and include them all in (30.1).

We have

$$I_g = \int \mathcal{L} d^4x, \qquad (30.1)$$

where this Lagrangian density $\mathcal{L}$ is $(16\pi)^{-1}$ times the $\mathcal{L}$ of Section 26. We get

$$\delta I_g = \int \left(\frac{\partial \mathcal{L}}{\partial g_{\alpha\beta}}\delta g_{\alpha\beta} + \frac{\partial \mathcal{L}}{\partial g_{\alpha\beta,\nu}}\delta g_{\alpha\beta,\nu}\right) d^4x$$

$$= \int \left[\frac{\partial \mathcal{L}}{\partial g_{\alpha\beta}} - \left(\frac{\partial \mathcal{L}}{\partial g_{\alpha\beta,\nu}}\right)_{,\nu}\right]\delta g_{\alpha\beta} d^4x \qquad \text{(Euler-Lagrange)}$$

The work of Section 26, leading to (26.11), shows that

$$\frac{\partial \mathcal{L}}{\partial g_{\alpha\beta}} - \left(\frac{\partial \mathcal{L}}{\partial g_{\alpha\beta,\nu}}\right)_{,\nu} = -\frac{1}{16\pi}(R^{\alpha\beta} - \tfrac{1}{2}g^{\alpha\beta}R)\sqrt{}. \qquad (30.2)$$

Let ϕ_n ($n = 1, 2, 3, \ldots$) denote the other field quantities. Each of them is assumed to b e a component of a tensor, but its precise tensor character is left unspecified. I' is of the form of the integral of a scalar density

$$I' = \int \mathcal{L}' d^4x,$$

where $\mathcal{L}'$ is a function of the ϕ_n and their first derivatives $\phi_{n,\mu}$ and possibly also higher derivatives.

The variation of the action now leads to a result of the form

$$\delta(I_g + I') = \int (p^{\mu\nu}\delta g_{\mu\nu} + \textstyle\sum_n \Phi^n \delta\phi_n)\sqrt{} d^4x, \qquad (30.3)$$

with $p^{\mu\nu} = p^{\nu\mu}$, because any term involving δ (derivative of a field quantity) can be transformed by partial integration to a term that can be included in (30.3). The variation principle (30.1) thus leads to the field equations

$$p^{\mu\nu} = 0, \qquad (30.4)$$
$$\Phi^n = 0. \qquad (30.5)$$

$p^{\mu\nu}$ will consist of the term (30.2) coming from I_g plus terms coming from L', say $N^{\mu\nu}$. We have of course $N^{\mu\nu} = N^{\nu\mu}$. L' usually does not contain derivatives of the fundamental tensor $g_{\mu\nu}$ and then

$$N^{\mu\nu} = \frac{\partial L}{\partial g_{\mu\nu}}. \qquad (30.6)$$

The equation (30.4) now becomes

$$R^{\mu\nu} - ½g^{\mu\nu}R - 16\pi N^{\mu\nu} = 0$$

It is just the Einstein equation (24.6) with

$$Y^{\mu\nu} = -2N^{\mu\nu}. \qquad (30.7)$$

We see here how each field contributes a term to the right-hand site of the Einstein equation, depending, according to (30.6), on the way the action for that field involves $g_{\mu\nu}$.

It is necessary for consistency that the $N^{\mu\nu}$ have the property $N^{\mu\nu}{}_{:\nu} = 0$. This property can be deduced quite generally from the condition that I' is invariant under a change of coordinates that leaves the bounding surface unchanged.

We make a small change of coordinates, say $x^{\mu'} = x^\mu + b^\mu$, with the b^μ small and functions of the x's, and work to the first order in the b^μ. The transformation law for the fundamental tensor ($g_{\mu\nu}$) is according to (3.7), with dashed suffixes to specify the new tensor,

$$g_{\mu\nu}(x) = x^{\alpha'}{}_{,\mu}x^{\beta'}{}_{,\nu}g_{\alpha'\beta'}(x') \qquad (30.8)$$

Let $\delta g_{\alpha\beta}$ denote the first-order change in $g_{\alpha\beta}$, not at a specified point, but for definite values of coordinates to which it refers, so that

$$g_{\alpha'\beta'}(x') = g_{\alpha\beta}(x') + \delta g_{\alpha\beta} = g_{\alpha\beta}(x) + g_{\alpha\beta,\sigma}b^\sigma + \delta g_{\alpha\beta}.$$

We have

$$x^{\alpha'}{}_{,\mu} = (x^\alpha + b^\alpha)_{,\mu} = g^\alpha{}_\mu + b^\alpha{}_{,\mu} .$$

Thus (30.8) gives

$$\begin{aligned} g_{\mu\nu}(x) &= (g^\alpha{}_\mu + b^\alpha{}_{,\mu})(g^\beta{}_\nu + b^\beta{}_{,\nu})[g_{\alpha\beta}(x) + g_{\alpha\beta,\sigma}b^\sigma + \delta g_{\alpha\beta}] \\ &= g_{\mu\nu}(x) + g_{\mu\nu,\sigma}b^\sigma + \delta g_{\mu\nu} + g_{\mu\beta}b^\beta{}_{,\nu} + g_{\alpha\nu}b^\alpha{}_{,\mu} \quad , \end{aligned}$$

so

$$\delta g_{\mu\nu} = -g_{\mu\alpha}b^\alpha{}_{,\nu} - g_{\nu\alpha}b^\alpha{}_{,\mu} - g_{\mu\nu,\sigma}b^\sigma .$$

We now determine the variation in I' when the $g_{\mu\nu}$ are changed in this way and the other field variables keep the same value at the point with coordinates $x^{\mu'}$ that they previously had for x^μ. It is, if we use (30.6),

$$
\begin{aligned}
\delta I' &= \int N^{\mu\nu}\,\delta g_{\mu\nu}\sqrt{}\ d^4x \\
&= \int N^{\mu\nu}(-g_{\mu\alpha}b^{\alpha}{}_{,\nu} - g_{\nu\alpha}b^{\alpha}{}_{,\mu} - g_{\mu\nu,\sigma}b^{\sigma})\sqrt{}\ d^4x \\
&= \int [2(N_{\alpha}{}^{\nu}\sqrt{})_{,\nu} - g_{\mu\nu,\alpha}N^{\mu\nu}\sqrt{}]b^{\alpha}d^4x \\
&= 2\int N_{\alpha}{}^{\nu}{}_{:\nu}\, b^{\alpha}\sqrt{}\ d^4x
\end{aligned}
$$

from the theorem expressed by (21.4), which is valid for any symmetrical two-index tensor. The invariance property of I' requires that it shall be unaltered under this variation, for all b^{α}.
Hence $N_{\alpha}{}^{\nu}{}_{:\nu} = 0$.
On account of this relation, the field equations (30.4) and (30.5) are not all independent.