The Hamilton Lecture:

Geometrization By Ricci Flow

Richard Hamilton

Sept. 12th, 2004

The Hamilton Lecture: Geometrization By Ricci Flow

Notes taken by **Xiaodong Cao**

Preface

This lecture note is based on Professor Richard Hamilton's lectures given at Columbia University when I was a Ritt Assistant Professor during 2002-2006. It is also based on my understanding to geometric evolution equations.

The goal for this lecture note is to keep writing of Professor Hamilton's lecture notes, especially for those parts which hadn't appeared elsewhere before. I apologize for not listing out all the references. The reader should check those recent papers/books for detailed reference.

And at last, this note is far from complete and is always under updating.

Contents

Chapter 1

Background on Ricci Flows

The structure of this chapter is organized as follows. In Section 1, we introduced un-normalized and normalized Ricci flow equations. Short-time existence was proved in Section 2. Some evolution equations for curvatures was derived in Section 3. In Section 4, a property of long time existence was discussed.

1 Ricci Flow Equation

Let (M, g) be a compact Riemannian manifold. We try to improve a Riemannian metric $g(X, Y)$ by evolving it by its Ricci curvature $Rc(X, Y)$. A family of Riemannian metric $g(t)$, $t \in [0, T)$, where $T \in (0, \infty]$, is called a solution to the *Ricci flow* if

$$\frac{\partial}{\partial t} g(x, t) = -2Rc(x, t) \tag{1.1}$$

at all points $x \in M$ and time $t \in [0, t)$. In other words, for any tangent vectors X and Y at x we have:

$$\frac{\partial}{\partial t} g(X, Y)(x, t) = -2Rc(X, Y)(x, t) \tag{1.2}$$

for all points $x \in M$ and time $t \in [0, t)$. This is a second-order weakly parabolic system. We usually write the equation in the component form

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} \tag{1.3}$$

Since the Ricci flow equation does not preserve the volume, we often consider the *normalized* Ricci flow defined by

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} + \frac{2}{n} r g_{ij} \tag{1.4}$$

where

$$r = \frac{\int R dV}{\int dV}$$

is the average scalar curvature.

2 Short Time Existence and Modified Ricci Flow

Since the Ricci flow equation is only weakly parabolic, so the first question is that of short-time existence. In [2], Hamilton proved that on a compact manifold, a solution exists for short time for any smooth initial metric, which made the study of Ricci flow possible. The proof was simplified by D. DeTurck[1] later.

Proposition 1 (Hamilton) *Given any smooth, compact Riemannian manifold* (M, g_0)*, there exists a unique solution* $g(t)$ *to the Ricci flow with initial condition* $g(0) = g_0$ *on some time interval* $[0, \epsilon)$*.*

DeTurck's idea is to modify the Ricci flow by a diffeomorphism induced by a time-dependent vector field V. First we have the following lemma:

Lemma 1 *Let $v(y,t)$ $(y \in M,\ t \in R^+)$ be a time-varying vector field on M. Then for small t, there exists a unique family of diffeomorphisms $\varphi_t : M \to M$ such that*

$$\frac{\partial}{\partial t}\varphi_t(x) = v(\varphi_t(x), t) \tag{2.1}$$

for all $x \in M$ and with $\varphi_0 = identity$

The standard proof when v does not depend on t still applies, via the existence and uniqueness theorem for ordinary differential equations.

For any tensor T, the time differential of T under the diffeomorphism is equal to $L_v T$, for the metric tensor g, we have

$$(L_v g)_{ij} = g_{ik} D_j v^k + g_{jk} D_i v^k$$

for a vector field $v = v^k \frac{\partial}{\partial x^k}$.

Consider the modified flow

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} + g_{ik} D_j v^k + g_{jk} D_i v^k$$

we want to choose v^k such that the equation is strictly parabolic.

2.1 Symbol

We can view symbol $\Sigma(\zeta)$ as a quadratic function on cotangent space $\zeta = \vartheta_i dx^i$. For

$$\frac{\partial}{\partial t} u = a^{ij} \frac{\partial^2 u}{\partial x^i \partial x^j} + b_i \frac{\partial u}{\partial x^i} + c$$

the symbol $\Sigma(\zeta) = a^{ij} \zeta_i \zeta_j$ We say that the partial differential equation is strictly parabolic if the symbol $\Sigma(\zeta)$ is positive definite, i.e., the matrix (a_{ij}) is positive definite.

In the case of constant coefficient case, under the space Fourier transformation for $\hat{u}(\zeta)$, we have

$$\frac{\partial}{\partial t}\hat{u}(\zeta) = -a^{ij}\zeta_i\zeta_j\hat{u}(\zeta) + \sqrt{-1}b_i\zeta_i\hat{u}(\zeta) + c\hat{u}(\zeta)$$

When the symbol is positive definite, the high frequences decays rapidly when ζ become large.

2.2 Quasi-linear

If the symbol of the partial differential equation

$$\frac{\partial}{\partial t}u = a^{ij}(x,t,u,\frac{\partial u}{\partial x^k})\frac{\partial^2 u}{\partial x^i \partial x^j} + b(x,t,u,\frac{\partial u}{\partial x^k})$$

is positive for all $\zeta \neq 0$, we say the partial differential equation is quasi-linear. The following theorem says that if the symbol is positive for initial data $u(0)$ then the quasi-linear partial differential equation has a short time solution.

2.3 Quasi-linear System

The Ricci flow equation is quasi-linear, the coefficient of $\frac{\partial^2 g_{kl}}{\partial x^i \partial x^j}$ is linear while the coefficient of $\frac{\partial g_{jk}}{\partial x^j}$ is quadratic. So now let's compute the symbol of $R_{ik} = g^{jl}R_{ijkl} = g^{jl}g_{kp}R^k_{ijl}$ in the following, we will omit the lower order by using $\cdots$ and only write those leading terms.

$$R_{ik} = g^{jl}g_{kp}\{\frac{\partial}{\partial x^i}\Gamma^k_{jl} - \frac{\partial}{\partial x^i j}\Gamma^k_{il} + \cdots\} = g^{jl}g_{kp}\{\frac{\partial}{\partial x^i}[\frac{1}{2}g^{pq}(g_{lq,j} + g_{jq,l} - g_{jl,q})] - sym(i,j) + \cdots\}$$

We can simplify this to

$$\frac{1}{2}g^{jl}\{[g_{jk,il} - g_{jl,ik}] - sym(i,j) + \ldots\}$$

So we have

$$\frac{\partial}{\partial t}g_{ik} = g^{jl}(g_{ik,jl} - g_{jk,il} - g_{il,jk} + g_{jl,ik}) + \cdots$$

De Turk's trick is to add $L_v g$ to the right hand side and make the modified flow equation a strictly parabolic one.

Lemma 2 *Let $g_{ij}(y,t)$ $(y \in M,\ t \in R^+)$ be a time-varying Riemannian metric on M, and φ_t the family of diffeomorphisms from above. Then*

$$\frac{\partial}{\partial t}\varphi_t^*(g)(x) = \varphi_t^*[\frac{\partial}{\partial t}g(\varphi_t(x))] + 2\varphi_t^*[\delta^*\omega(\varphi_t(x))] \tag{2.2}$$

where ω is the one-form $\omega_i = g_{ik}v^k$.

The proof is by direct computation. So (1.3) has a solution. This solution is also unique(see [2] or [4]).

3 Evolution of Curvature

When the Riemannian metric evolves, so does its curvature. Under the un-normalized Ricci flow, Hamilton[2] has proved that the curvatures evolve as follows:

Proposition 2 *Under un-normalized Ricci flow equation, the curvature tensor $R_{ijkl}(t)$, the Ricci curvature tensor $R_{ij}(t)$ and the scalar curvature $R(t)$ satisfy the following* *evolution equations*

$$\begin{aligned}\frac{\partial}{\partial t}R_{ijkl} =& \triangle R_{ijkl} + 2(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}) \\ & - g^{pq}(R_{pjkl}R_{qi} + R_{ipkl}R_{qj} + R_{ijpl}R_{qk} + R_{ijkp}R_{ql}) \quad (3.1)\\ \frac{\partial}{\partial t}R_{ik} =& \triangle R_{ik} + 2g^{pa}g^{qb}R_{piqk}R_{ab} - 2g^{pq}R_{pi}R_{qk} \quad (3.2)\\ \frac{\partial}{\partial t}R =& \Delta R + 2g^{ij}g^{kl}R_{ik}R_{jl} \quad (3.3)\end{aligned}$$

The proof is by direct calculation (see [2]). Since the term $g^{ij}g^{kl}R_{ik}R_{jl}$ is just the squared norm of the Ricci curvature, which is always positive, now by using the maximal principle, we have:

Corollary 1 *If the scalar curvature $R \geq C > 0$ at time $t = 0$, then it remains so under the Ricci flow.*

This is an example of the Ricci flow "prefers" positive curvature. A very important question in the study of Ricci flow is: *Which properties of curvature may be preserved by Ricci flow equation?*

A consequence result of the above is that the un-normalized Ricci flow (1.2) has to end in finite time if the compact manifold has positive scalar curvature everywhere at the initial time.

Corollary 2 *If $R(0) \geq c > 0$, then $T \leq n/2c$, here T is the maximum existence time of Ricci flow Eq.(1.3).*

Proof. Set

$$f(x,t) = nc/(n-2ct),$$

so

$$\frac{\partial}{\partial t}f = 2f^2/n$$

thus

$$\frac{\partial}{\partial t}(R-f) \geq \triangle(R-f) + \frac{2}{n}(R+f)(R-f)$$

since

$$|R_{ij} - \frac{R}{n}g_{ij}|^2 \geq 0$$

implies

$$g^{ij}g^{kl}R_{ik}R_{jl} \geq R^2/n.$$

As

$$R - f \geq 0$$

at $t=0$, this property remains true under Ricci flow on $[0,T)$, but $f(x,t) \to \infty$ when $t \to n/2c$, so $T \leq n/2c$. q.e.d.

4 Long Time Existence

On a compact manifold of any dimension with any given initial metric at $t=0$, we have

Proposition 3 (Hamilton) *For any smooth initial metric on a compact manifold there exists a maximal time T on which there is a unique smooth solution to the Ricci flow for $0 \leq t \leq T$. Either $T = \infty$ or else the curvature is unbounded as $t \to T$.*

Proof. We follow Hamilton's proof in [2], the proof is by contradiction. Since we already know short time existence and uniqueness, we can take the maximum time interval $0 \leq t < T$ on which the solution exists. We will show that if $T < \infty$ and $|Rm| \leq C$ as $t \to T$, then the metric g_{ij} converges as $t \to T$ to a limit metric (which is strictly positive definite), and all the derivatives converge also, showing the limit metric is smooth. We could then use the short time existence result to continue the solution past T, showing T is not maximal. The following 3 lemmas are needed for the proof:

Lemma 3 *Let g_{ij} be a time-dependent metric on M for $0 \le t < T \le \infty$. Suppose*

$$\int_{t=0}^{T} \max_M |g'_{ij}| dt \le C < \infty$$

then the metric $g_{ij}(t)$ for all different time are equivalent, and they converge as $t \to T$ uniformly to a positive-definite metric tensor $g_{ij}(T)$ which is continuous and also equivalent.

Lemma 4 *If $|Rm| \le C$ on $0 \le t < T < \infty$, then for any n we can find a constant C_n with*

$$\int_M |\partial^n Rm|^2 d\mu \le C_n$$

Lemma 5 *If $|Rm| \le C_0$ on $0 \le t < T < \infty$, then $|\partial^n Rm| \le C_n$ for all n. The constant C_n depends only on the initial value of the metric and the constant C_0.*

The estimates on $Rc = R_{ij}$ follows from those on Rm. Since $\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$, it's easy to see that $g_{ij}(t)$ have all their derivatives bounded, and converge to the limit metric $g_{ij}(T)$ in the C^∞ topology as $t \to T$. This completes the proof of the Proposition 3. q.e.d.

By using the derivative estimate in [4], Hamilton gives a similar proof for this result.

Chapter 2

Soliton and Perelman's Estimate

The structure of this chapter is organized as follows. In Section 1, we talk about Ricci solitons. In section 2, we begin Perelman's estimate. In section 3, we introduce the logarithmic sobolev inequality. In section 4, we continue to introduce Perelman's W-functional. In section **??**, we give a quick review of injectivity radius. In section 5, we talk about the injectivity radius estimate. In section **??**, we study the no breathers theorem. In section **??**, we study the no local collapsing theorem.

1 Soliton

Soliton Equation:

$$\frac{\partial}{\partial t} g_{ij} = (L_v g)_{ij}$$

If $v = -Df$, this is called gradient soliton. Under Ricci flow, we have the steady gradient Ricci soliton equation:

$$R_{ij} = D^2_{ij} f$$

Some calculation shows that

$$D_i R + 2R_{ij} D_j f = 0$$

Hence on the gradient soliton we have

$$D_i(R + |Df|^2) = D_i R + 2R_{ij} D_j f = 0$$

Lemma 6 *On a steady gradient Ricci soliton,*

$$R + |Df|^2 = M$$

Here M is a space constant.

More calculation shows that

$$\frac{\partial}{\partial t}(D_i D_j f) = \triangle(D_i D_j f) + 2R_{ikjl} D_k D_l f - R_{il} D_j D_l f - R_{jl} D_i D_l f$$

At the same time, we already have the evolution equation for Ricci curvature:

$$\frac{\partial}{\partial t} R_{ij} = \triangle R_{ij} + 2R_{ikjl} R_{kl} - 2R_{ik} R_{jk}$$

Hence we have the following:

Lemma 7

$$D_i D_j f = R_{ij} \Leftrightarrow D_i D_j (\frac{\partial}{\partial t} - \triangle) f = 0$$

So on a gradient Ricci soliton,

$$\frac{\partial}{\partial t} f = \triangle f + C(t)$$

And later we will see that $C(t) = -M$.

Perelman introduced the following equation (not only on a soliton):

$$\frac{\partial}{\partial t} f = -\triangle f - |Df|^2 + R$$

In fact, on a gradient Ricci soliton, $R = \triangle f$, so when $C(t) = -M$, the above two equations are same.

2 Perelman's Estimate I

Perelman adjoins function f as above to go with the Ricci flow equation to form a system.

$$\begin{cases} \frac{\partial}{\partial t} g_{ij} = -2R_{ij} \\ \frac{\partial}{\partial t} f = -\triangle f - |Df|^2 + R \end{cases}$$

Suppose the Ricci flow exists for $t \in [0, T]$, then specify initial data for g_{ij} at $t = 0$ and final data for f at $t = T$.

Remark. 1) The time T here is strictly less the blow-up time we discussed before.
2) The Ricci flow equation is a forward "heat" equation, and the equation about f is a backward heat equation.
3) f_{max} decreases backward.

Lemma 8

$$\frac{d}{dt} \int e^f dV = 0$$

So if we define

$$W = \int (|Df|^2 + R) e^f dV$$

We have the following:

Lemma 9 *(Perelman) Under Ricci flow, W is non-decreasing, in fact,*

$$\frac{d}{dt}W = 2\int |D_iD_jf - R_{ij}|^2 e^f dV \geq 0$$

Proof.

$$\begin{aligned}\frac{d}{dt}W =& \frac{d}{dt}\int (g^{ij}D_ifD_jf + R)e^f dV \\ =& \int [2g^{ik}g^{jl}R_{kl}D_ifD_jf + 2g^{ij}D_ifD_j(\frac{\partial}{\partial t}f) + \triangle R \\ &+ 2|R_{ij}|^2 + (|Df|^2 + R)(\frac{\partial}{\partial t}f - R)]e^f dV \\ =& \int [2(\triangle f)^2 + 3\triangle f|Df|^2 + |Df|^4 - 2R\triangle f - 2R|Df|^2 \\ &+ 2R_{ij}D_ifD_jf + 2|R_{ij}|^2]e^f dV\end{aligned}$$

q.e.d.

3 Logarithmic Sobolev Inequalities

$$\int_M f^2 \log f \rho d\mu - \int_M f^2 \rho d\mu \log(\int_M f^2 \rho d\mu)^{1/2} \leq \int_M (|\nabla^M f|^2 + Vf^2)\rho d\mu$$

Or in other form,

$$\int_M f^2 \log f d\mu \leq \int_M |\nabla f|^2 d\mu$$

Where $d\mu = (2\pi)^{-\frac{n}{2}} e^{-\frac{x^2}{2}} dx$ and $\int_M f^2 d\mu = 1$.

4 Perelman's Estimate II

We now begin to deal with the shrinking case. (Shrinking soliton)

Choose any $T > 0$, look at $0 \le t < T < \infty$. Define

$$W = (T-t)^{-\frac{n}{2}} \int [(T-t)(|Df|^2 + R) - f]e^f dV$$

and let's change the evolution equation of f:

$$\frac{\partial}{\partial t} f = -\triangle f - |Df|^2 + R - \frac{n}{2(T-t)}$$

then we have

$$(T-t)^{-n/2} \int e^f dV = constant$$

Lemma 10

$$\frac{d}{dt} W = 2(T-t)^{1-n/2} \int |D_i D_j f - R_{ij} + \frac{1}{2(T-t)} g_{ij}|^2 e^f dV$$

Remark. Here T is just a parameter, does not need to be the blow-up time. (If we divide by T, we should get the first formula?)

Proof. In this proof, let's put a "$^-$" on f and W, because we want to use the formula we derived before. And let f and W be as in section 2,

Let $\tau = T - t$, $\bar{f} = f + \frac{n}{2} \log \tau$, $Y = \int f e^f dV$, we have $D\bar{f} = Df$, and

$$\tau^{-n/2} \int e^{\bar{f}} dV = \int e^f dV = constant$$

$$\frac{\partial}{\partial t} \bar{f} = -\triangle \bar{f} - |D\bar{f}|^2 + R - \frac{n}{2(T-t)}$$

$$\bar{W} =\tau \int (|Df|^2 + R)e^f dV - \int (f + \frac{n}{2}\log\tau)e^f dV \tag{4.1}$$
$$=\tau W - Y - \frac{n}{2}\log\tau \int e^f dV \tag{4.2}$$

We also need the following

$$\frac{d}{dt}Y =\frac{d}{dt}(\int fe^f dV) \tag{4.3}$$
$$= \int [(f+1)e^f \frac{\partial}{\partial t}f - Rfe^f]dV \tag{4.4}$$
$$= \int \{(f+1)e^f[-\triangle f - |Df|^2 + R] - Rfe^f\}dV \tag{4.5}$$
$$= \int fe^f(-\triangle f - |Df|^2)dV + \int e^f(-\triangle f - |Df|^2 + R)dV \tag{4.6}$$
$$= \int f(-\triangle(e^f))dV + \int (-\triangle(e^f) + Re^f)dV \tag{4.7}$$
$$= \int e^f|Df|^2 dV + \int Re^f dV \tag{4.8}$$

Here we have used

$$\triangle e^f = D(e^f Df) = e^f(\triangle f + |Df|^2)$$

and integration by parts. We also have

$$\int (\triangle f)e^f dV = -\int |Df|^2 e^f dV$$

Now we can compute $\frac{d}{dt}\bar{W}$:

$$\frac{d}{dt}\bar{W} = \tau 2\int |D_iD_jf - R_{ij}|^2 e^f dV - \int (|Df|^2 + R)e^f dV - \frac{d}{dt}Y + \frac{n}{2\tau}\int e^f dV \quad (4.9)$$

$$= 2\tau\int |D_iD_jf - R_{ij}|^2 e^f dV - 2\int (|Df|^2 + R)e^f dV + \frac{n}{2\tau}\int e^f dV \quad (4.10)$$

$$= 2\tau\int |D_iD_jf - R_{ij}|^2 e^f dV + 2\int (\triangle f - R)e^f dV + \frac{n}{2\tau}\int e^f dV \quad (4.11)$$

$$= 2\tau\int [|D_iD_jf - R_{ij}|^2 + \frac{1}{\tau}(\triangle f - R) + \frac{n}{4\tau^2}]e^f dV \quad (4.12)$$

$$= 2\tau\int |D_iD_jf - R_{ij} + \frac{1}{2(T-t)}g_{ij}|^2 e^f dV \quad (4.13)$$

q.e.d.

We now no longer keep the "$^-$" on f and W.

Corollary 3 *W is non-decreasing under the Ricci flow.*

In next section, we want to estimate the lower bound on the injectivity radius.

5 Injectivity Radius Estimate

The reason we want the injectivity radius estimate is that if $\rho \ll \frac{1}{\sqrt{K_{max}}}$, then if we choose the point where the injectivity radius is very small as the origin, dilate in the space such that the injectivity radius keep fixed as 1, and take the limit as time approach the blow-up time, then we will get a flat manifold with some little loop, i.e., $B_r^{n-1} \times S_\epsilon^1$. Where B_r^{n-1} is the ball of radius r and S_ϵ^1 is the circle with radius ϵ.

Assume we have a ball of radius r around some point P, where $|Rm| \leq \frac{1}{r^2}$ and $V(B_r(P)) \leq \varepsilon r^n$, if ε is too small, we can conclude that W is very negative, this contradicts the fact that W is bounded from below.

First let us change the notation, let

$$e^f = u^2$$

i.e.

$$u = e^{f/2},\ f = 2\log u,\ Df = 2\frac{Du}{u}$$

so

$$\tau^{-n/2}\int e^f dV = \tau^{-n/2}\int u^2 dV$$

and

$$W = \tau^{-n/2}\int [\tau(|Df|^2 + R) - f]e^f dV \tag{5.1}$$

$$= 4\tau^{-n/2}\int [\tau(|Du|^2 + \frac{1}{4}Ru^2) - \frac{1}{2}u^2 \log u]dV \tag{5.2}$$

Chapter 3

Curvature pinching

The structure of this chapter is organized as follows. In Section 1, by using the maximal principle for system, we can study the reaction ODE system instead of the PDE system.. In section 2, we derive the pinching estimate for three-manifolds. In section **??**, we study when the curvature condition is preserved. In section 3, we derive the pinching estimate for positive Ricci curvature. In section 4, we study the general curvature pinching condition.

1 Maximal Principle for System

Maximal principle proved to be a useful tool in Ricci flow. We need to establish the following:

Proposition 4 (Hamilton) *If a tensor F evolves by a diffusion-reaction equation*

$$\frac{\partial}{\partial t}F = \triangle F + \Phi(F) \tag{1.1}$$

and if Z is a closed subset of the tensor bundle which is invariant under parallel translation and such that its intersection with each fibre is convex, and if Z is preserved by the system of ordinary equations in each fibre given by the reaction

$$\frac{d}{dt}F = \Phi(F) \tag{1.2}$$

then Z is also preserved by the diffusion-reaction equation, in the sense that if the tensor lies in Z at each point at the start, then it continues to lie in Z subsequently. For preserving curvature inequality in the Ricci flow we take Z to be a subset of the bundle of curvature operators M which is convex in each fibre, and check that Z is preserved by the curvature reaction

$$\frac{d}{dt}M = M^2 + M^{\#} \tag{1.3}$$

Proof. See [3] q.e.d.

The importance of above Proposition is that it allows us to study the reaction ODE system instead of the PDE system.

2 Pinching Estimates for 3-manifolds

Let's consider the Einstein tensor $E_{ij} = \frac{1}{2}Rg_{ij} - R_{ij}$. Eigenvalues of E_{ij} are principal sectional curvature. In an orthonormal frame $\{F_1, F_2, F_3\}$ which diagonalize the matrix E_{ij} , E_{11} is the sectional curvature of the plane spanned by $\{F_2, F_3\}$, i.e., $E_{11} = R_{2323}$. If we pick an orientation which gives the volume form u_{ijk}, then

$$R_{ijkl} = E_{pq}g^{pr}g^{qs}u_{ijr}u_{kls}$$

We also know that in the orthonormal frame, R_{ij} is also diagonal. Hence if we have

$$E_{ij} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix}$$

$$R_{ij} = \begin{pmatrix} \mu+\nu & 0 & 0 \\ 0 & \lambda+\nu & 0 \\ 0 & 0 & \lambda+\mu \end{pmatrix}$$

and finally, $R = 2(\lambda + \mu + \nu)$ is the scalar curvature.

Now let's consider the evolving orthonormal frame defined as before

$$F_a = F_a^i \frac{\partial}{\partial x^i}$$

$$\frac{\partial}{\partial t} F_a^i = g^{ij} R_{jk} F_a^k$$

And the pull back metric $g_{ab} = g_{ij} F_a^i F_b^j$ is constant in time. We have

$$\frac{\partial}{\partial t} E_{ab} = \triangle E_{ab} + 2(E_{ab}^2 + E_{ab}^{\#})$$

if

$$E_{ab} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix}$$

then

$$E^2_{ab} = \begin{pmatrix} \lambda^2 & 0 & 0 \\ 0 & \mu^2 & 0 \\ 0 & 0 & \nu^2 \end{pmatrix}$$

and

$$E^{\#}_{ab} = \begin{pmatrix} \mu\nu & 0 & 0 \\ 0 & \lambda\nu & 0 \\ 0 & 0 & \lambda\mu \end{pmatrix}$$

The equation

$$\frac{d}{dt}E_{ab} = 2(E^2_{ab} + E^{\#}_{ab})$$

gives us a system of ODEs on 6-dimensional space of symmetric matrices. And we are looking for convex set $Z \subseteq \mathbb{R}^6$ preserved by ODE, and hence by the PDE.

Remark. 1) Since we define the convex set Z by the eigenvalues of matrix (or any other tensors), it will be invariant under parallel translation, so we only need to verify it's convex and preserved by the ODEs.
2) The ODEs preserve the diagonal form, so we only need to consider on the subset of diagonal matrices.

And now the reaction equation for E_{ab} (in the space of symmetric 3×3 matrices) descends to the reaction on the diagonal terms (α, β, γ) in R^3 given by the system of ODE:

$$\tfrac{1}{2}\tfrac{d}{dt}\lambda = \lambda^2 + \mu\nu$$
$$\tfrac{1}{2}\tfrac{d}{dt}\mu = \mu^2 + \lambda\nu$$
$$\tfrac{1}{2}\tfrac{d}{dt}\nu = \nu^2 + \lambda\mu$$

Let's assume that $\lambda \geq \mu \geq \nu$, and this order is preserved by ODE. We have the following:

Lemma 11 *The functions* $\lambda, \lambda+\mu, \lambda+\mu+\nu$ *are convex functions. The functions* $\nu, \mu+\nu, \lambda+\mu+\nu$ *are concave functions.*

3 Pinching Estimate for Positive Ricci Curvature

Assume that M^3 is compact with $Ric > 0$, that means

$$\mu+\nu > 0$$

on M. So

$$\mu+\nu \geq \epsilon > 0$$

for some $\epsilon > 0$ at $t = 0$.

Let's define the pinching set

$$Z = \{E_{ab} : \lambda - \nu \leq A(\mu+\nu)^{1-\delta} \text{ and } \lambda+\mu \leq B(\mu+\nu)\}$$

for some constants A and B we will choose below, we can always choose A, δ and B such that this is true for any initial metric on compact 3-manifolds with positive Ricci curvature, then for the fixed δ, we can choose $A < \infty$ such that this holds everywhere and at any time $t > 0$.

Proposition 5 *On a compact 3-manifold with positive Ricci curvature, we have*

$$\lambda - \nu \leq A(\mu+\nu)^{1-\delta}$$

for some constants A and δ, hence if λ is very big, then

$$\frac{\lambda - \nu}{\lambda} \ll 1$$

Now the following

Lemma 12 *If $R_{ij} = f g_{ij}$ for some scalar function f and dimension is larger than 2, then f must be a constant. (For example, see Gallot, P. 109.)*

says that around where λ (or R) is very big, it must be constant curvature. Notice the lemma is a local argument. Now by using the derivative estimate on curvature, the curvature can't fall off rapidly, and by the continuity argument, it must be constant curvature everywhere. (? but that's on limit, before limit, why is it even true for locally to be a constant? Now on the limit, it's constant, so before the limit, it's almost constant in a small neighborhood, then apply the derivative estimate and using the continuity argument.)

4 General Curvature Pinching Condition

Conjecture 1 *For $n > 4$, there exists $\epsilon > 0$ such that*

$$R_{ijkl}\varphi_{ij}\varphi_{kl} \geq \epsilon R|\varphi|^2$$

is preserved by the Ricci flow.

Conjecture 2 *The positive Ricci curvature is preserved by the $K\ddot{a}hler - Ricci$ flow.*

Remark. Because of Perelman's estimate, the type I singularity of $K\ddot{a}hler - Ricci$-flow must be compact. (Injectivity radius and volume estimate lead to the bound on the diameter.)

Chapter 4

Ricci Soliton

The structure of this chapter is organized as follows. In Section 1, we talk about maximal principle. In section 2, we present some Harnack inequalities. In section 3, we study gradient Ricci soliton. In section 4, we study eternal solutions.

1 Strong Maximal Principle

Theorem 1 *If*

$$\frac{\partial}{\partial t} f \geq \triangle f \tag{1.1}$$

on M^n *(need not to be complete) for* $0 \leq t \leq T$*, if*

$$f \geq 0$$

everywhere and

$$f(P, 0) > 0$$

for some point P*, then*

$$f(Q, t) > 0$$

for all Q *and* $t > 0$.

2 Harnack Inequality

If a solution to a parabolic partial differential equation is positive in some sense, then we can try to derive so called "Harnack Inequality (Estimate)", in the form of

$$\frac{\partial}{\partial t} Max \geq -c$$

where $-c$ is any lower bound.

Example.

$$f(x,t) = ce^{-n^2 t} \sin(nx)$$

is a solution of

$$\frac{\partial}{\partial t} f = \frac{\partial^2 f}{\partial x^2}$$

$$f_{max} = ce^{-n^2 t}$$

$$\frac{d}{dt} f_{max} = -n^2 f_{max}$$

Example.

$$f = \frac{c}{\sqrt{t}} e^{-\frac{x^2}{4t}}$$

is a positive solution of the equation

$$\frac{\partial}{\partial t} f = \frac{\partial^2 f}{\partial x^2}$$

$$f_{max} = \frac{c}{\sqrt{t}}$$

$$\frac{d}{dt} f_{max} = -\frac{c}{2t^{3/2}} = -\frac{f_{max}}{2t}$$

Corollary 4 *Any non-negative solution of*

$$\frac{\partial}{\partial t} f = \frac{\partial^2 f}{\partial x^2}$$

satisfies the following inequality:

$$\frac{\partial}{\partial t} f_{max} \geq -\frac{f_{max}}{2t}$$

for $t \geq 0$.

Theorem 2 *If f is any positive solution of*

$$\frac{\partial}{\partial t} f = \frac{\partial^2 f}{\partial x^2}$$

then

$$\frac{\partial}{\partial t} f + \frac{1}{2t} f \geq \frac{1}{f} (\frac{\partial}{\partial x} f)^2$$

Proof. Let

$$h = \frac{\partial}{\partial t} f + \frac{1}{2t} f - \frac{1}{f} (\frac{\partial}{\partial x} f)^2$$

q.e.d.

For curve-shrinking flow, we have

$$\frac{\partial}{\partial t} k + \frac{1}{2t} k \geq \frac{1}{k} (\frac{\partial k}{\partial s}$$

For Ricci flow, we have

$$\frac{\partial}{\partial t}R + \frac{1}{t}R \geq \frac{1}{R}|DR|^2$$

For mean curvature flow, see

For gauss curvature flow, see Chow.

For Yamabe flow, see Chow.

Theorem 3 *Suppose we have non-negative curvature operator, then*

$$\frac{\partial}{\partial t}R + \frac{1}{t}R \geq R_{ij}^{-1}D_iRD_jR$$

3 Gradient Ricci Soliton

In chapter **??**, we know that a steady gradient Ricci Soliton satisfies:

$$D_iD_jf = R_{ij} \tag{3.1}$$

while the expanding gradient Ricci Soliton satisfies:

$$D_iD_jf = R_{ij} + rg_{ij}$$

Remark. If $r < 0$, it's shrinking, if $r > 0$, it's expanding.

Question: Are all solitons gradient?

Answer: See Perelman [5].

Let us look at steady soliton, if we take derivative and anti-symmetry, we have

$$
\begin{aligned}
D_i R_{jk} - D_j R_{ik} =& D_i D_j D_k f - D_j D_i D_k f \\
=& R_{ijkl} D_l f
\end{aligned} \tag{3.2}
$$

On soliton, let us define

$$
P_{ijk} = D_i R_{jk} - D_j R_{ik} = R_{ijkl} D_l f
$$

contract (j, k) we have

$$
D_i R - \frac{1}{2} D_i R = -R_{il} D_l f
$$

or

$$
D_i R + 2 R_{ij} D_j f = 0 \tag{3.3}
$$

Now on a soliton,

$$
D_i (R + |Df|^2) = D_i R + 2 D_j f D_i D_j f = D_i R + 2 D_j f \cdot R_{ij} = 0
$$

Hence $R + |Df|^2 = M$ is a constant on soliton. This is Lemma 6 in Chapter **??**.

For Kähler-Ricci flow, pick f, such that

$$
D_\alpha D_{\tilde{\beta}} f = R_{\alpha \tilde{\beta}}
$$

and

$$
D_\alpha D_\beta f = 0 \equiv R_{\alpha\beta}
$$

For mean curvature flow:

$$H^2 + |Df|^2 = constant$$

However, there is no equality known for Yamabe flow.

Now, we take another derivative on both sides of Eq 3.2:

$$\begin{aligned} D_iD_jR_{kl} - D_iD_kR_{jl} = & \quad D_iR_{jklm}D_mf + R_{jklm}D_iD_mf \\ = & \quad D_iR_{jklm}D_mf + R_{jklm}R_{im} \end{aligned} \tag{3.4}$$

Contract (i,j)

$$\triangle R_{kl} - D_iD_kR_{il} = D_iR_{iklm}D_mf + R_{im}R_{iklm} \tag{3.5}$$

while

$$\begin{aligned} D_iD_kR_{il} = & \quad D_kD_iR_{il} + R_{ikim}R_{lm} + R_{iklm}R_{im} \\ = & \quad \tfrac{1}{2}D_kD_iR + R_{km}R_{lm} - R_{kilm}R_{im} \end{aligned}$$

and

$$\begin{aligned} D_iR_{iklm} = & \quad D_iR_{lmik} \\ = & \quad -D_lR_{miik} - D_mR_{ilik} \\ = & \quad D_lR_{mk} - D_mR_{lk} \end{aligned}$$

The second equality is by 2nd Bianchi identity. So Eq. 3.4 becomes:

$$\triangle R_{kl} - \frac{1}{2} D_k D_l R + 2R_{mn} R_{kmln} - R_{km} R_{lm}$$
$$= (D_l R_{mk} - D_m R_{lk} D_m f \tag{3.6}$$

Define

$$M_{kl} = \triangle R_{kl} - \frac{1}{2} D_k D_l R + 2R_{mn} R_{kmln} - R_{km} R_{lm}$$

Then on a soliton

$$M_{kl} = P_{lmk} D_m f$$

Since $P_{lmk} = -P_{mlk}$, $M_{kl} + P_{mlk} D_m f = 0$, we also have

$$P_{mlk} D_m f + R_{kmln} D_m f D_n f = 0$$

so on a soliton

$$M_{kl} + 2P_{mlk} D_m f + R_{kmln} D_m f D_n f = 0 \tag{3.7}$$

For any one-form W,

$$M_{kl} W_k W_l + 2P_{mlk} W_k W_l + R_{kmln} D_m f D_n f W_k W_l = 0 \tag{3.8}$$

If we anti-symmetrize, let $U_{km} = \frac{1}{2}(W_k D_m f - W_m D_k f)$ then

$$M_{kl} W_k W_l + 2P_{mlk} U_{ml} W_k + R_{kmln} U_{km} U_{ln} = 0 \tag{3.9}$$

we can prove the following theorem:

Theorem 4 *Given solution to the Ricci Flow, complete with bounded curvature, and* *with positive curvature operator, i.e.,*

$$R_{abcd}U_{ab}U_{cd} \geq 0$$

for all 2 forms U_{ab}. If it's an expanding soliton:

$$R_{ij} = D_i D_j f + \frac{c}{t} g_{ij}$$

Then:

$$Z = (M_{ab} + \frac{1}{2t} R_{ab} W_a W_b + 2P_{abc} U_{ab} W_c + R_{abcd} U_{ab} U_{cd} \geq 0$$

for all W_a, U_{ab} on soliton: $U = W \wedge Df$. W and U on soliton has special relation.

Proof. $Z > 0$, small $t > 0$. maximal principal.

if $Z = 0$ somewhere, find first time t, point p, W, U where $z = 0$. extend W, U to no bound. at point p,

$$\begin{cases} D_a U_{bc} = \frac{1}{2}(R_{ab} W_c - R_{ac} W b) + \frac{1}{4t}(g_{ab} W_c - g_{ac} W_b \\ D_a W_b = 0 \end{cases}$$

$$(D_t - \triangle) W_a = \frac{1}{t} W_a, (D_t - \triangle) U_{ab} = 0$$

(only at p, require anti-symmetric of b and c)

$$\begin{aligned} (D_t - \triangle) Z =& (P_{abc} W_c + R_{abcd} U_{cd})(P_{abe} W_e + R_{abef} W_e W_f) \\ &+ 2R_{abcd} M_{cd} W_a W_b - 2P_{acd} W_a W_b + 8R_{abce} P_{dbe} U_{ab} W_c \\ &+ 4R_{aecf} R_{bedf} U_{ab} U_{cd} \geq 0 \end{aligned}$$

at non-vector.

Write

$$Z = \sum_{\mu} \lambda_{\mu}[(V_{\mu}, U) + (X_{\mu}, W)]^2$$

then

$$\begin{aligned}(D_t - \triangle)Z =& | \sum_{\mu} \lambda_{\mu}((V_{\mu}, U) + (X_{\mu}, W))V_m|^2 \\ &+ \sum_{\mu,N} \lambda_M \lambda_N([V_M, V_N], \nu)^2 \\ &+ \sum_{M,N} (V_M \lrcorner X_N - V_N \lrcorner X_M, W)^2 \\ \geq& 0\end{aligned}$$

q.e.d.

$$\begin{cases} M : X \times X \\ P : X \times V \\ R_{abcd} : V \times V \end{cases}$$

Trace Z, write $U_{ab} = \frac{1}{2}(-W_a V_b + W_b V_a) = V \wedge W$, $Z =$ quadratic in W in each V. trace W, $\sum$ orthonormal basis of W's.

$$\begin{aligned}M_{ab} =& \triangle R_{ab} - \frac{1}{2} D_a D_b R \\ &+ 2R_{acbd}R_{cd} - R_{ac}R_{bc} + \frac{1}{2t} R_{ab} W_a W_b\end{aligned}$$

trace $= \triangle R - \frac{1}{2}\triangle R + R_{cd}R_{cd} + \frac{1}{2t} R$

trace $M = \frac{1}{2}\triangle R + |R_{iC}|^2 + \frac{1}{2t}R$,

$$P_{abc}U_{ab}W_c = (D_aR_{bc} - D_bR_{ac})V_aW_bW_c = +\frac{1}{2}D_bR \cdot V_b$$

trace $R_{abcd}V_aW_bV_cW_d = R_{ac}V_aV_c$,

$$\frac{1}{2t}R + |R_{ic}|^2 + +\frac{1}{2}\triangle R + D_aRV_a + R_{ab}V_aV_b \geq 0$$

all V.

$$\frac{1}{2t}R + |R_{ic}|^2 + 2D_aRV_a + 2R_{ab}V_aV_b \geq 0$$

$$\frac{\partial}{\partial t}Rt + \frac{1}{2t}R + 2D_aRV_a + 2R + abV_aV_b \geq 0$$

all V.

inf_v:

$$D_aR + 2R_{ab}V_b = 0$$

$$V_b = -\frac{1}{2}R_{ab}^{-1}D_aR$$

Hence

$$\frac{\partial}{\partial t}Rt + \frac{1}{t}R \geq \frac{1}{2}R_{ab}^{-1}D_aRD_bR$$

when $n = 2$, $R_{ab} = \frac{1}{2}Rg_{ab}$, so $R_{ab}^{-1} = 2\frac{1}{R}g_{ab}^{-1}$

This is the trace form of Harnack Inequality.

Conjecture 3 *Is the Harnack Inequality true for Ric > 0 in dim 3?*

Remark. Positive Sectional curvature $\Longrightarrow$ Positive Ricci curvature.

In Kähler case, if the manifold has positive holomorphic bisectional curvature (which is preserved by Ricci flow), i.e.,

$$R_{\alpha\bar{\beta}\bar{z}\bar{\delta}} W^{\alpha} Z^{\bar{\beta}} W^{\bar{z}} Z^{\delta} \geq 0, \forall W, Z.$$

$$M_{\alpha\bar{\beta}} = \frac{1}{2} \triangle R_{\alpha\bar{\beta}}$$

$$D_{\alpha} R_{\beta\bar{z}} = D_{\beta} R_{\alpha\bar{z}}$$

$$\begin{aligned} D_{\alpha} D_{\bar{\beta}} =& D_{\alpha} D_{\bar{\beta}} R_{z\bar{z}} \\ =& D_{\alpha} D_{\bar{z}} R_{z\bar{\beta}} \\ =& D_{\zeta} D_{\bar{z}} R_{\alpha\bar{\beta}} \\ =& \triangle R_{\alpha\bar{\beta}} \end{aligned}$$

Hence the Harnack Inequality is:

$$\frac{\partial}{\partial t} R_{\alpha\bar{\beta}} t + \ \geq 0$$

4 Eternal Solutions

Eternal Solution with bounded curvature are important because they occur as models of slowly forming singularities.

Theorem 5 *Suppose we have a complete solution with bounded curvature to the Ricci flow with (weakly) positive curvature operator for all time, $-\infty < t < \infty$, and suppose R attains its max at some point p, for time t. Then the solution is a Ricci gradient soliton, i.e., $\exists f$ with $R_{ij} = D_i D_j f$.*

In dimension 3, if we have a type 2 solution, then the limit satisfies $-\infty < t < \infty$, $R(0,0) = 1$, $R(x,t) < 1$ and $R_m \geq 0$ by the pinching estimate.

Corollary 5 *In dimension 3, any type 2 blow up must be a soliton.*

This is also true for dimension 4 when the metric has positive isotropic curvature.

Notice if the solution is ancient (or etenal), then the $\frac{1}{t}$ term in the Harnack inequality will drop, because we can start at any negative time $-c$ and that term will become $\frac{1}{t+c}$, let $c- >\to \infty$. Hence the Harnack inequality for ancient solution is

$$\frac{\partial}{\partial t}R + 2D_aRV_a + 2R_{ab}V_aV_b \geq 0$$

Corollary 6 *On ancient solution with non-negative curvature operator, R increases everywhere.*

Proof. $\frac{\partial}{\partial t}R \geq \frac{1}{2}R_{ab}^{-1}D_aRD_bR > 0$ q.e.d.

Chapter 5

The L-function and Harnack Estimate

The structure of this chapter is organized as follows. In Section 1, we talk about Perelman's L-function. In section 2, we present Li-Yau's Harnack estimate. In section 3, we derive Perelman's Harnack inequality in a different way.

1 the L-function

Suppose we have a solution to the Ricci flow

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$$

on M for $0 \leq t \leq T$.

Let $\tau = T - t$, $\frac{\partial}{\partial \tau} = -\frac{\partial}{\partial t}$. In space-time,

Define

$$\mathcal{L}(\gamma) = \int_{\tau_1}^{\tau_2} \sqrt{\tau}(R + g_{ij}\frac{dx^i}{d\tau} \cdot \frac{dx^j}{d\tau})d\tau$$

$$L((P,\tau_1),(Q,\tau_2)) = inf_\gamma \mathcal{L}(\gamma)$$

Let us take a 1-parameter family of paths, with parameter u. i.e.,

$$\begin{aligned} x^i =& x^i(\tau,u) \\ X^i =& \frac{\partial}{\partial \tau} x^i \\ Y^i =& \frac{\partial}{\partial u} x^i \end{aligned}$$

X,Y on the tangent space are space like.

along path

$$\frac{d}{d\tau} = \frac{\partial}{\partial \tau} + \frac{\partial x^i}{\partial \tau} \cdot \frac{\partial}{\partial x^i}$$

In the following, we will compute $D_i L$,$D_i D_j L$ and then trace of $D^2 L = \triangle L$

$$\begin{aligned} D_\tau X =& (\frac{\partial^2 x^i}{\partial \tau^2} + \Gamma^i_{jk} \frac{\partial x^j}{\partial \tau}\frac{\partial x^k}{\partial \tau})\frac{\partial}{\partial x^i} \\ D_\tau Y =& (\frac{\partial^2 x^i}{\partial \tau \partial u} + \Gamma^i_{jk} \frac{\partial x^j}{\partial \tau}\frac{\partial x^k}{\partial u})\frac{\partial}{\partial x^i} \end{aligned}$$

Let $\mathcal{L} = \mathcal{L}(u) = \int_{\tau_1}^{\tau_2} \sqrt{\tau}(R + g_{ij}\frac{\partial x^i}{\partial \tau}\frac{\partial x^j}{\partial \tau})d\tau$

$$\frac{d\mathcal{L}}{du} = \int_{\tau_1}^{\tau_2} \sqrt{\tau}(\frac{\partial R}{\partial x^k}\frac{\partial x^k}{\partial u} + 2g_{ij}\frac{\partial^2 x^i}{\partial \tau \partial u} \cdot \frac{\partial x^j}{\partial \tau} + \frac{\partial g_{ij}}{\partial x^k}\frac{\partial x^k}{\partial u}\frac{\partial x^i}{\partial \tau}\frac{\partial x^j}{\partial \tau})d\tau \tag{1.1}$$

If we fix the end points

$$\begin{aligned} 0 = \sqrt{\tau} g_{ij}\frac{\partial x^i}{\partial u}\frac{\partial x^j}{\partial \tau}|_{\tau 1}^{\tau 2} =& \int_{\tau_1}^{\tau_2} \frac{d}{d\tau}(\sqrt{\tau}(g_{ij}\frac{\partial x^i}{\partial u}\frac{\partial x^j}{\partial \tau}))d\tau \\ = \int_{\tau_1}^{\tau_2} \sqrt{\tau}(g_{ij} & \frac{\partial^2 x^i}{\partial \tau \partial u}\frac{\partial x^j}{\partial \tau} + g_{ij}\frac{\partial x^i}{\partial u}\frac{\partial^2 x^j}{\partial \tau^2} + 2R_{ij}\frac{\partial x^i}{\partial u}\frac{\partial x^j}{\partial \tau} \\ & + \frac{\partial g_{ij}}{\partial x^k}\frac{\partial x^k}{\partial \tau}\frac{\partial x^i}{\partial u}\frac{\partial x^j}{\partial \tau} + \frac{1}{2\tau} g_{ij}\frac{\partial x^i}{\partial u}\frac{\partial x^j}{\partial \tau})d\tau \end{aligned} \tag{1.2}$$

Here, we assumed the following: $x^i(\tau_1,u)=p^i$ and $x^i(\tau_2,u)=q^i$ or $\frac{\partial x^i}{\partial u}=0$ at τ_1,τ_2.

Otherwise, we will have a boundary term:

$$2\sqrt{\bar{\tau}} < X, Y >$$

If $Y \neq 0$ at $\tau_2 = \bar{\tau}$, i.e., we only fix the initial point.

(1.1) − 2(1.2), we have

$$\frac{d\mathcal{L}}{du} = \int_{\tau_1}^{\tau_2} \sqrt{\tau}\frac{\partial x^k}{\partial u}[\frac{\partial R}{\partial x^k} - 2g_{kj}(\frac{\partial^2 x^j}{\partial \tau^2} + \Gamma^j_{il}\frac{\partial x^i}{\partial \tau}\frac{\partial x^l}{\partial \tau}) - 4R_{jk}\frac{\partial x^j}{\partial \tau} - \frac{1}{\partial \tau}g_{jk}\frac{\partial x^j}{\partial \tau}]d\tau \quad (1.3)$$

The advantage of local coordinates is $\frac{\partial}{\partial x^i}$,$\frac{\partial}{\partial \tau}$,$\frac{\partial}{\partial u}$ just act as ordinary derivative, but we need to keep in mind that for the result we need to put into a form tensor-like, so that the result is coordinate independent.

And we can rewrite (3) as

$$\frac{d\mathcal{L}}{du} = -\int_{\tau_1}^{\tau_2} 2\sqrt{\tau}\frac{\partial x^k}{\partial u}g_{jk}[(\frac{\partial^2 x^j}{\partial \tau^2} + \Gamma^j_{il}\frac{\partial x^i}{\partial \tau}\frac{\partial x^l}{\partial \tau}) + 2g^{kl}R_{lm}\frac{\partial x^m}{\partial \tau} - \frac{1}{2}g^{jl}D_l R + \frac{1}{\partial \tau}\frac{\partial x^j}{\partial \tau}]d\tau \quad (1.4)$$

when it is a minimal path $\frac{d\mathcal{L}}{du}=0$ for all $\frac{\partial x^k}{\partial u}=0$ at τ_1,τ_2.

So we have the following lemma:

Lemma 13 *The equation for $\mathcal{L}$-minimizing path is*

$$\frac{\partial^2 x^j}{\partial \tau^2} + \Gamma^j_{il}\frac{\partial x^i}{\partial \tau}\frac{\partial x^l}{\partial \tau} + 2g^{jl}R_{lm}\frac{\partial x^m}{\partial \tau} - \frac{1}{2}g^{jl}D_l R + \frac{1}{2\tau}\frac{\partial x^j}{\partial \tau} = 0 \quad (1.5)$$

From now on, we ignore all Γ's or $\frac{\partial}{\partial x}g$.

Next we compute the second variation formula for $\mathcal{L}$:

$$\frac{d^2\mathcal{L}}{du^2} = \int_{\tau_1}^{\tau_2} \sqrt{\tau}(\frac{\partial^2 R}{\partial x^k \partial x^l} \cdot \frac{\partial x^k}{\partial u} \cdot \frac{\partial x^l}{\partial u} + \frac{\partial R}{\partial x^k}\frac{\partial^2 x^k}{\partial u^2} + 2g_{ij}\frac{\partial^3 x^i}{\partial\tau\partial u^2}\frac{\partial x^j}{\partial\tau}$$
$$+2g_{ij}\frac{\partial^2 x^i}{\partial\tau\partial u}\frac{\partial^2 x^j}{\partial\tau\partial u} + \frac{\partial^2 g_{ij}}{\partial x^k\partial x^i} \cdot \frac{\partial x^k}{\partial u}\frac{\partial x^l}{\partial u}\frac{\partial x^i}{\partial\tau}\frac{\partial x^j}{\partial\tau})d\tau \qquad (1.6)$$

Now we only fix the start point. Let $\tau_1 = 0$, $\tau_2 = \bar{\tau}$.

$$2\sqrt{\bar{\tau}} < D_Y Y, X >|_{end} = \int_0^{\bar{\tau}} \frac{d}{d\tau}\{2\sqrt{\tau}g_{ij}(\frac{\partial^2 x^i}{\partial u^2} + \Gamma^i_{kl}\frac{\partial x^k}{\partial u}\frac{\partial x^l}{\partial u})\frac{\partial x^j}{\partial\tau}\}$$
$$= \int_0^{\bar{\tau}} \{2\sqrt{\tau}g_{ij}\frac{\partial^3 x^i}{\partial\tau\partial u^2}\frac{\partial x^j}{\partial\tau} + 2\sqrt{\tau}(\frac{\partial}{\partial x^k}R_{jl} + \frac{\partial}{\partial x^l}R_{jk} - \frac{\partial}{\partial x^j}R_{kl})\frac{\partial x^k}{\partial u}\frac{\partial x^l}{\partial u}\frac{\partial x^j}{\partial u}$$
$$+2\sqrt{\tau}g_{ij}(\frac{\partial}{\partial x^m}\Gamma^i_{kl})\frac{\partial x^m}{\partial\tau}\frac{\partial x^k}{\partial u}\frac{\partial x^l}{\partial u}\frac{\partial x^j}{\partial\tau} + 2\sqrt{\tau}g_{ij}\frac{\partial^2 x^i}{\partial u^2}\frac{\partial^2 x^i}{\partial\tau^2}$$
$$+4\sqrt{\tau}R_{ij}\frac{\partial^2 x^i}{\partial u^2} + \frac{1}{\sqrt{\tau}}g_{ij}\frac{\partial^2 x^i}{\partial u^2}\frac{\partial x^j}{\partial\tau}\}d\tau$$

Here we used

$$\frac{d}{d\tau}\Gamma^i_{kl} = \frac{\partial}{\partial\tau}\Gamma^i_{kl} + \frac{\partial}{\partial x^m}\Gamma^i_{kl} \cdot \frac{\partial x^m}{\partial\tau}$$

and

$$\frac{\partial}{\partial\tau}\Gamma^i_{kl} = \frac{\partial}{\partial\tau}\{\frac{1}{2}g^{ij}\{\frac{\partial}{\partial x^k}g_{il} + \frac{\partial}{\partial x^l}g_{jk} - \frac{\partial}{\partial x^j}g_{lk}\}\}$$
$$= g^{ij}\{\frac{\partial}{\partial x^k}R_{jl} + \frac{\partial}{\partial x^l}R_{jk} - \frac{\partial}{\partial x^j}R_{lk}\}$$

so

$$\frac{d^2\mathcal{L}}{du^2} = 2\sqrt{\bar{\tau}} < D_Y Y, X > + \int_0^{\bar{\tau}} [D_k D_l R Y^k Y^l + 2g_{ij}\frac{\partial^2 x^j}{\partial\tau\partial u} \cdot \frac{\partial^2 x^j}{\partial\tau\partial u}] +$$
$$\sqrt{\bar{\tau}}\{-\frac{\partial^2}{\partial x^k\partial x^m}g_{jl} - \frac{\partial^2}{\partial x^m\partial x^l}g_{jk} + \frac{\partial^2}{\partial x^m\partial x^j}g_{kl} + \frac{\partial^2}{\partial x^k\partial x^l}\}\frac{\partial x^j}{\partial\tau}\frac{\partial x^m}{\partial\tau}\frac{\partial x^k}{\partial u}\frac{\partial x^l}{\partial u}$$

Now let us choose $Y = \{\frac{\partial x^i}{\partial u}\}$ to solve the first order ODE on $[0, \bar{\tau}]$:

$$\frac{\partial^2 x^i}{\partial \tau \partial u} + \Gamma^i_{jk} \frac{\partial x^j}{\partial \tau} \frac{\partial x^k}{\partial u} + g^{ij} R_{jk} \frac{\partial x^k}{\partial u} - \frac{1}{2\tau} \frac{\partial x^j}{\partial u} = 0$$

Lemma 14 *If Y solve the above ODE and $|Y(\bar{\tau})| = 1$, then $|Y|^2 = \frac{\tau}{\bar{\tau}}$.*

Proof.

$$\begin{aligned} \frac{d}{d\tau}|Y|^2 =& \frac{d}{d\tau}\{g_{ij} \frac{\partial x^i}{\partial u} \frac{\partial x^j}{\partial u}\} \\ =& 2g_{ij} \frac{\partial^2 x^i}{\partial \tau \partial u} \frac{\partial x^j}{\partial u} + 2R_{ij} \frac{\partial x^i}{\partial u} \frac{\partial x^j}{\partial u} \\ =& \frac{1}{\tau} g_{ij} \frac{\partial x^i}{\partial u} \frac{\partial x^j}{\partial u} = \frac{1}{\tau}|Y|^2 \end{aligned}$$

so: $\frac{d}{d\tau}(\frac{|Y|^2}{\tau}) = 0$. Hence $|Y|^2 = \frac{\tau}{\bar{\tau}}$. q.e.d.

Remark. we can see from above that $Y \to 0$ as $\tau \to 0$.

So we have

$$\begin{aligned} Hess_L(Y, Y) =& \frac{d^2 \mathcal{L}}{du^2} - 2\sqrt{\bar{\tau}} < D_Y Y, X > \\ =& \frac{1}{\sqrt{\bar{\tau}}} - 2\sqrt{\bar{\tau}} Ric(Y, Y) - \int_0^{\bar{\tau}} \sqrt{\tau} H(X, Y) d\tau \end{aligned}$$

Here

$$H(X, Y) = 2[M_{ij} Y^i Y^j - P_{ijk} Y^i X^j Y^k + R_{ijkl} X^i X^j X^k X^l]$$

$$M_{ij} = \triangle R_{ij} - \frac{1}{2} D_i D_j R + 2R_{ikjl} R_{kl} - R_{ik} R_{jk} - \frac{1}{2\tau} R_{ij}$$

$$P_{ijk} = D_i R_{jk} - D_j R_{ik}$$

Since we have fixed the start point P, $\tau_1 = 0$. Let

$$L(Q,\bar{\tau}) = inf_{\gamma}\mathcal{L}((P,0),(Q,\bar{\tau}))$$

We first verify that for a gradient shrinking soliton, if a point flow along Df, then the result path is the "L-geodesic". Since

$$R_{ij} = -D_iD_jf + \frac{1}{2\tau}g_{ij}$$

"L-geodesic" attains $min\mathcal{L}$ solves the second order ODE:

$$\left(\frac{d^2x^i}{d\tau^2} + \Gamma^i_{jk}\frac{dx^j}{d\tau}\frac{dx^k}{d\tau}\right) - \frac{1}{2}g^{ij}D_jR + \frac{1}{2\tau}\frac{dx^i}{d\tau} + 2g^{ij}R_{jk}\frac{dx^k}{d\tau} = 0$$

$$\frac{dx^i}{d\tau} = \frac{\partial x^i}{\partial \tau} = g^{ij}D_jf$$

$$\frac{\partial f}{\partial \tau} = -g^{ij}D_ifD_jf$$

$$\frac{df}{d\tau} = \frac{\partial f}{\partial \tau} + \frac{\partial f}{\partial x^i}\cdot\frac{dx^i}{d\tau} = 0$$

$$D_hR_{ij} - D_iR_{hj} = -R_{hijk}g^{kl}D_lf$$

trace by g^{ij}:

$$D_hR - \frac{1}{2}D_hR = R_{hk}g^{kl}D_lf$$

i.e., $D_hR = 2R_{hk}g^{kl}D_lf$.

So,

$$\begin{aligned}
\frac{d^2x^i}{d\tau^2} + \Gamma^i_{jk}\frac{dx^j}{d\tau}\cdot\frac{dx^k}{d\tau} &= \frac{d}{d\tau}(g^{ij}D_jf) + \Gamma^i_{jk}\frac{dx^j}{d\tau}\cdot\frac{dx^k}{d\tau}\\
&= \frac{\partial}{\partial\tau}(g^{ij}D_jf) + \frac{\partial}{\partial x^k}(g^{ij}D_jf)\cdot\frac{\partial x^k}{\partial\tau}\\
&= -2g^{il}R_{lk}g^{kj}D_jf + g^{ij}D_j[-g^{kl}D_kfD_lf] + g^{ij}\frac{\partial^2 f}{\partial x^k\partial x^j}\frac{\partial x^k}{\partial\tau}\\
&= -2g^{ij}R_{jk}\frac{dx^k}{d\tau} - 2g^{ij}g^{kl}D_jD_kf\cdot D_lf + g^{ij}\frac{\partial^2 f}{\partial x^k\partial x^j}\frac{\partial x^k}{\partial\tau}
\end{aligned}$$

(again, we ignore $\Gamma's$ in calculation)

So

$$\begin{aligned}
&\frac{d^2x^i}{d\tau^2} + \Gamma^i_{jk}\frac{dx^j}{d\tau}\cdot\frac{dx^k}{d\tau} - \frac{1}{2}g^{ij}D_jR + \frac{1}{2\tau}\frac{dx^i}{d\tau} + 2g^{ij}R_{jk}\frac{dx^k}{d\tau}\\
&= -2g^{ij}R_{jk}\frac{dx^k}{d\tau} - 2g^{ij}g^{kl}D_jD_kf\cdot D_lf + g^{ij}\frac{\partial^2 f}{\partial x^k\partial x^j}\frac{\partial x^k}{\partial\tau}\\
&- \frac{1}{2}g^{ij}D_jR + \frac{1}{2\tau}\cdot g^{ij}D_jf + 2g^{ij}R_{jk}\frac{dx^k}{d\tau}\\
&= -2g^{ij}R_{jk}\frac{dx^k}{d\tau} - 2g^{ij}g^{kl}(-R_{jk} + \frac{1}{2\tau}g_{jk})D_lf\\
&+ g^{ij}(-R_{kj} + \frac{1}{2\tau}g_{kj})\frac{\partial x^k}{\partial\tau} - \frac{1}{2}g^{ij}D_jR + \frac{1}{2\tau}g^{ij}D_jf + 2g^{ij}R_{jk}\frac{dx^k}{d\tau}\\
&= 2g^{ij}g^{kl}R_{jk}D_lf - g^{ij}R_{kj}\frac{\partial x^k}{\partial\tau} - \frac{1}{2}g^{ij}D_jR\\
&= 2g^{ij}R_{jk}D_lf - g^{ij}R_{kj}D_kf - g^{ij}R_{jk}D_kf\\
&= 0
\end{aligned}$$

We used $D_hR = 2R_{hk}g^{kl}D_lf$ in the last step. So we have proved the following:

Corollary 7 *The gradient flow of f actually is a L-geodesic.*

Remark. For those who are more familiar with Hamilton's notation, this might be a little confusing, but always remember that Perelman's potential function is the negative of Hamilton's.

2 Li-Yau's Harnack Estimate

Let us first recall Li-Yau's Harnack Estimate: Let $u > 0$ be a solution of the heat equation.

$$u_t = u_{xx}$$

take $u = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}$ be the fundamental solution.

Let $F = \frac{x^2}{4t}$, so $u = \frac{1}{(4\pi t)^{1/2}} e^{-F}$

$$\frac{\partial u}{\partial t} = \frac{1}{(4\pi t)^{1/2}} e^{-F} [-\frac{\partial F}{\partial t} - \frac{1}{\partial t}]$$

$$u_{xx} = \frac{1}{(4\pi t)^{1/2}} e^{-F} [-F_{xx} + |DF|^2]$$

so

$$\frac{\partial F}{\partial t} = F_{xx} - |DF|^2] - \frac{1}{2t}.$$

Let $H = F_{xx} - \frac{1}{2t}$. This is our Harnack quantity. Harnack vanishes on Foundamental solution. Since

$$F_{xx} = \frac{1}{2t}.$$

We have

$$H_t = F_{xxt} + \frac{1}{2t^2} = (F_{xx} - |DF|^2 - \frac{1}{2t})_{xx} + \frac{1}{2t^2} = F_{xxxx} - 2F_x F_{xxx} - 2F_{xx}^2 + \frac{1}{2t^2}$$

$$H_x = F_{xxx}$$

$$H_{xx} = F_{xxxx}$$

so

$$H_t = H_{xx} - 2F_x \cdot H_x - 2(H + \frac{1}{2t})^2 + \frac{1}{2t^2}$$

i.e.

$$H_t = H_{xx} - 2F_x \cdot H_x - 2H^2 + \frac{2H}{t}$$

For any finite solution F, as $t \to 0$, $H \to -\infty$, so by maximal principle, $H \leq 0$ for all $t > 0$. i.e., on fundamental solution (which is not finite), $H = 0$, otherwise $H \leq 0$. So

$$F_{xx} \leq \frac{1}{2t}.$$

$$F_t = F_{xx} - F_x^2 = \frac{1}{2t}$$

$$H = F_{xx} - \frac{1}{2t} = F_t + F_x^2 \leq 0$$

$$\begin{aligned}
\frac{dF}{dt} &= F_t + F_x \cdot \frac{dx}{dt} \\
&\leq -F_x^2 + F_x \cdot \frac{dx}{dt} \\
&= -(F_x - \frac{1}{2}\frac{dx}{dt})^2 + \frac{1}{4}(\frac{dx}{dt})^2
\end{aligned}$$

So along any path:

$$\frac{dF}{dt} \leq \frac{1}{4}(\frac{dx}{dt})^2$$

Hence $\forall x_1,\, x_2,\, t_2 > t_1$

$$F(x_2,t_2) - F(x_1,t_1) \leq \frac{1}{4}\int_{t_1}^{t_2} (\frac{dx}{dt})^2 dt$$

Among all possible path

$$\int (\frac{dx}{dt})^2 dt = min \Longleftrightarrow \frac{d^2x}{dt^2} = 0$$

i.e., the path is straight line.

$$\frac{dx}{dt} = \frac{x_2 - x_1}{t_2 - t_1}$$

so

$$F(x_2,t_2) \leq F(x_1,t_1) + \frac{1}{4}\frac{(x_2 - x_1)^2}{t_2 - t_1}$$

so $u = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}$ satisfies

$$u(x_2,t_2) \geq \frac{t_2}{t_1}^{1/2} e^{-(x_2-x_1)^2/4(t_2-t_1)} u(x_1,t_1)$$

Remark. there exists a suitable choice of center of the fundamental solution, such that the above quality holds.

3 Perelman's Harnack Inequality

Suppose we are on a gradient shrinking Ricci soliton, as before, we take a fixed vector bundle and evolve a moving orthonormal frame

$$F^a = F^i_a \frac{\partial}{\partial x^i}$$

by

$$\frac{\partial}{\partial t} = g^{ij} R_{ij} F_a^k$$

and we have

$$R_{ab} + D_a D_b F = \rho g_{ab}$$

here

$$\rho = \rho(t)$$

so

$$D_a R_{bc} + D_a D_b D_c F = 0$$

commute (a, b) and contract (a, c), we have

$$D_a R = 2 R_{ab} D_b F$$

we also have:

$$\begin{aligned}
(D_t - \Delta) R_{ab} =& 2 R_{acbd} R_{cd} \\
(D_t - \Delta) D_a D_b F =& D_a D_b (D_t - \Delta) F + 2 R_{acbd} F_{cd} \\
(D_t - \Delta) \rho g_{ab} =& (\frac{\partial}{\partial t} \rho) g_{ab}
\end{aligned}$$

hence

$$D_a D_b (D_t - \Delta) F + 2 R_{acbd} (\rho g_{cd}) + (\frac{\partial \rho}{\partial t}) g_{ab} = 0$$

i.e.

$$D_a D_b [(D_t - \Delta) F - 2 \rho F] = (\frac{\partial \rho}{\partial t} - 2\rho^2) g_{ab}$$

hence

$$(D_t - \Delta)F = 2\rho F + b(t)$$

and $\frac{\partial \rho}{\partial t} = 2\rho^2$

so $\rho = \frac{1}{2(F-t)}$, since as $t \longrightarrow T$ shrinking soliton will blow-up at every point. or $\rho = \frac{1}{2\tau}$.

This is because $\Delta[(D_t - \Delta)F - 2\rho F] = (\frac{\partial \rho}{\partial t} - 2\rho^2)n$ on a compact manifold, the right hand side must be 0.

On the other hand, derive as before, we also have

$$D_a[R + |DF|^2 - \frac{1}{\tau}F] = D_a R + 2D_a D_b F \cdot D_b F - \frac{1}{\tau}F_a = 0$$

i.e.

$$R + |DF|^2 - \frac{F}{\tau} = C(t)$$

where $C(t)$ is a constant in space. We can pick F, such that this constant $C(t) = 0$.

Remark. we will see later that the choice of this constant makes difference.

so $R + |DF|^2 = \frac{1}{\tau}F$

Next we find the value of $b(t)$

$$\begin{aligned}(D_t - \Delta)R =& 2|R_c|^2 \\ (D_t - \Delta)|\nabla F|^2 =& 2D_a F \cdot D_a[(D_t - \Delta)F] - 2|D_a D_b F|^2 \\ (D_t - \Delta)(\frac{1}{\tau}F) =& \frac{1}{\tau^2}F + \frac{1}{\tau}(D_t - \Delta)F\end{aligned}$$

So

$$0 = 2|R_c|^2 - 2|D_a D_b F|^2 + 2D_a F \cdot D_a(2\rho F) - \frac{1}{\tau^2}F - \frac{1}{\tau}(2\rho F + b)$$

$$2\rho^2 \cdot n - 4\rho\Delta F + 2 \cdot 2 \cdot \rho|\nabla F|^2 - \frac{1}{\tau^2}F - \frac{2\rho F}{\tau} - \frac{b}{\tau} = 0$$

$$R + \Delta F = n\rho = \frac{n}{2\tau}$$

$$R + |DF|^2 = \frac{1}{\tau}F$$

so

$$\begin{aligned} 2\rho^2 \cdot n + 4\rho(\frac{1}{\tau}F - n\rho) - \frac{1}{\tau^2}F - \frac{F}{\tau^2} - \frac{b}{\tau} &= 0 \\ -\frac{b}{\tau} + 2n\rho^2 + \frac{2}{\tau^2}F - \frac{n}{\tau^2} - \frac{2F}{\tau^2} &= 0 \end{aligned}$$

So

$$\frac{b}{\tau} = \frac{n}{2\tau^2} - \frac{n}{\tau^2} = -\frac{n}{2\tau^2}$$

$$b = -\frac{n}{2\tau}$$

$$(D_t - \Delta)F = \frac{1}{\tau}F - \frac{n}{2\tau}$$

$$\begin{aligned} \frac{\partial}{\partial t}F =& \Delta F + \frac{1}{\tau}F - \frac{n}{2\tau} \\ =& |DF|^2 + \frac{n}{2\tau} - \frac{1}{\tau}F + \frac{1}{\tau}F - \frac{n}{2\tau} = |DF|^2 \end{aligned}$$

i.e. F moves by its own gradient flow!

We can write in another form

$$-\frac{\partial F}{\partial t} = \frac{\partial F}{\partial \tau} = \Delta F - |DF|^2 + R - \frac{n}{2\tau}$$

now let $u = (4\pi\tau)^{-\frac{n}{2}}e^{-F}$, easy to verify that u satisfies the following adjoint forward heat equation:

$$\frac{\partial u}{\partial \tau} = \Delta u - Ru$$

i.e., if v satisfies $\frac{\partial v}{\partial t} = \Delta v$, then

$$\int_M uv = constant$$

Let $H = 2\Delta F - |DF|^2 + R + \frac{1}{\tau}F - \frac{n}{\tau}$, then on gradient shrinking soliton, $H \equiv 0$.

Theorem 6 *(Perelman's Harnack) If u is any positive solution to the adjacent heat equation*

$$\frac{\partial u}{\partial \tau} = \Delta u - Ru$$

suppose

$$u = (4\pi\tau)^{-\frac{n}{2}}e^{-F}$$

and

$$H = 2\Delta F - |DF|^2 + R + \frac{1}{\tau}F - \frac{n}{\tau}$$

then

$$\frac{\partial H}{\partial \tau} + 2DF \cdot DH = \Delta H - \frac{1}{\tau}H - 2|R_{ab} + D_aD_bF - \frac{1}{2\tau}g_{ab}|^2$$

moreover $max(\tau H)$ is decreasing as τ increase.

Notice that if u is like a δ-function, i.e., $F \sim \frac{|x|^2}{4\tau}$. or if u is smooth, $F = log\frac{1}{(4\pi\tau)^{\frac{n}{2}}u}$, as $\tau \to 0$, $F \to +\infty$. So we can not apply maximal principle directly to H. Nevertheless, we have

$$\frac{\partial}{\partial \tau}(\tau H) + 2DF \cdot D(\tau H) = \Delta(\tau H) - 2\tau|R_{ab} + D_aD_bF - \frac{1}{2\tau}g_{ab}|^2$$

hence we have the above theorem.

If $u \to \delta(\rho, 0)$, as $\tau \to 0$. Since

$$u = (4\pi\tau)^{-\frac{n}{2}} e^{-F}$$

$F \approx \frac{|x|^2}{4\tau}$, $D_a F \approx \frac{x^a}{2\tau}$, $\Delta F \approx \frac{n}{2\tau}$

$$H \approx \frac{n}{\tau} - \frac{|x|^2}{4\tau^2} + R + \frac{|x|^2}{4\tau^2} - \frac{n}{\tau} \to R$$

which is finite.

So $\tau H \to 0$ as $\tau \to 0$

by the above theorem, τH should be negative for all τ. so $H \leq 0$, but this is only true for the positive solution whose initial data is heat kernel, remember that Li-Yau's harnack is true for any positive solution.

Now let

$$\Box = \frac{\partial}{\partial t} - \Delta$$

$$\Box^* = \frac{\partial}{\partial \tau} - \Delta + R$$

be its conjugate operator. Since

$$\int (\Box u) \cdot v = \int u(\Box^* v)$$

to verify the above, consider

$$\frac{d}{dt} \int uv = \int (\frac{d}{dt} u) v + u(\frac{d}{dt} v) + uv(-R)$$

so

$$\int (\frac{\partial}{\partial t} - \Delta)u \cdot v = \int -u\frac{\partial}{\partial t}v + R \cdot u \cdot v - u \cdot \Delta v = \int u(\frac{\partial}{\partial \tau} - \Delta + R)v$$

Now suppose h is a positive solution to

$$\Box h = 0$$

Let $v = \tau H u$

Lemma 15

$$\Box^* v = -2\tau u|R_{ij} + D_i D_j f - \frac{1}{2\tau}g_{ij}|^2 \leq 0$$

Proof.

$$\begin{aligned}
\Box^* v =& (\frac{\partial}{\partial \tau} - \Delta + R)(\tau H u) \\
=& u\frac{\partial}{\partial \tau}(\tau H) + (\frac{\partial}{\partial \tau}u) \cdot \tau H - u\Delta(\tau H) - \tau H \Delta u - \nabla(\tau H) \cdot \nabla u + R\tau H u \\
=& u(\frac{\partial}{\partial \tau}(\tau H) - \Delta(\tau H)) + \tau H(\frac{\partial}{\partial \tau}u - \Delta u) + R\tau H u - \nabla(\tau H) \cdot \nabla u \\
=& u(-2\nabla F \cdot \nabla(\tau H) - 2\tau|R_{ij} + D_i D_j f - \frac{1}{2\tau}g_{ij}|^2) + \tau H(-Ru) + R\tau H u - \nabla(\tau H) \cdot \nabla u \\
=& -2\tau u|R_{ij} + D_i D_j f - \frac{1}{2\tau}g_{ij}|^2
\end{aligned}$$

Here we used $\nabla u = -u \cdot \nabla F$ q.e.d.

We have:

Lemma 16 $\frac{d}{dt}\int hu = 0$ *and* $\frac{d}{dt}\int hv \geq 0$

Proof.

$$\begin{aligned}\frac{d}{dt}\int hu &= \int \frac{\partial h}{\partial t}\cdot u - h\cdot\frac{\partial u}{\partial \tau} - Rhu\\ &= \int \Delta h\cdot u - h\cdot\Delta u + Rhu - Rhu = 0\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}\int hv &= \int \frac{\partial h}{\partial t}\cdot v - h\cdot\frac{\partial v}{\partial \tau} - Rhv\\ &\geq \int \Delta h\cdot v - h\cdot\Delta v + Rhv - Rhv = 0\end{aligned}$$

i.e., as τ decreasing, t increasing. $\int hu$ is a constant while $\int hv$ is increasing. q.e.d.

Now as $\tau \to 0$, if $u \to \delta - function$, $\int hv \to 0$. So $\int hv \leq 0$ for any τ and any $h > 0$. So $v \leq 0$, for all $t < T$.

Corollary 8 *If* $u \to \delta_p$ *as* $\tau \to 0$.

$$H = 2\Delta F - |DF|^2 + R + \frac{1}{\tau}F - \frac{n}{\tau} = 2\frac{\partial F}{\partial \tau} + |DF|^2 - R + \frac{1}{\tau}F \leq 0$$

Along any space-time path $x = x(\tau)$.

$$\frac{d}{d\tau}F = \frac{\partial F}{\partial \tau} + DF\cdot\frac{dx}{d\tau}$$

$$
\begin{aligned}
\frac{d}{d\tau}(2\sqrt{\tau}F) =& 2\sqrt{\tau}\cdot\frac{\partial F}{\partial\tau} + 2\sqrt{\tau}DF\cdot\frac{dx}{d\tau} + \frac{1}{\sqrt{\tau}}F \\
\leq& \sqrt{\tau}(-|DF|^2 + R - \frac{1}{\tau}F + 2DF\cdot\frac{dx}{d\tau} + \frac{1}{\tau}F) \\
=& \sqrt{\tau}(-|DF|^2 + 2DF\cdot\frac{dx}{d\tau}) + R\sqrt{\tau} \\
=& \sqrt{\tau}(-|DF - \frac{dx}{d\tau}|^2) + \sqrt{\tau}(R + (\frac{dx}{d\tau})^2) \\
\leq& \sqrt{\tau}(R + (\frac{dx}{d\tau})^2)
\end{aligned}
$$

Integrate from a point p and $\tau = 0$ to a point q and $\bar{\tau}$ along a path $x = x(\tau)$ where as $\tau \to 0$

$$x = x(\tau) \approx 2\sqrt{\tau}V + \cdots (higher\ \ order)$$

$F \approx \frac{|x|^2}{4\tau} = |v|^2$ and $\sqrt{\tau}F \to 0$

we have

$$2\sqrt{\bar{\tau}}F \leq \int_0^{\bar{\tau}} \sqrt{\tau}(R + |\frac{dx}{d\tau}|^2)d\tau = L(q,\bar{\tau})$$

as defined in Perelman's paper [5]. Recall

$$l = \frac{1}{2\sqrt{\tau}}L$$

Corollary 9

$$F \leq l$$

We also have a nice lower bound on heat kernel:

Corollary 10

$$u = (4\pi\tau)^{-\frac{n}{2}}e^{-F} \geq (4\pi\tau)^{-\frac{n}{2}}e^{-l}$$

Remark. If injectivity radius $\to 0$, these estimates are not good, because all heat kernels are summing up. (?)

Using the above, we can also derive the entropy estimate as a corollary:

$$W = \int \tau^{-\frac{n}{2}}[\tau(R + |DF|^2 + F - n)e^{-F}]dV = \int vdV$$

Notice that

$$\int \tau He^{-F}dV = \int [\tau(|DF|^2 + R) + F - n]e^{-F}dV$$

and

$$\int \Delta F \cdot e^{-F}dV = \int |DF|^2 \cdot e^{-F}dV$$

$$\Box^* v = -2\tau u|R_{ij} + D_iD_jF - \frac{1}{2\tau}g_{ij}|^2$$

where

$$v = \tau Hu = \tau^{\frac{-n}{2}}e^{-F}(\tau H)$$

we get the above identity via integration by parts. Now we state the entropy estimate as a corollary:

Corollary 11

$$\frac{dW}{d\tau} = \int (\frac{\partial v}{\partial \tau} + Rv)dV \leq \int (\Delta v)dV = 0$$

Chapter 6

Ancient Solutions

1 Basic Properties

Suppose we have a solution to the Ricci flow

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$$

which exists on $-\infty < t < T \leq \infty$. We call this an ancient solution.

Remark. Not all solution can be ancient solution.

Example. The fundamental solution to the heat equation

$$u_t = u_{xx}$$
$$u = \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4t}}$$

can only back to $t = 0$.

And we have the following theorem

Theorem 7 *If u is an ancient solution for* $u_t = u_{xx}$ *on* $-\infty < t < T \leq \infty$ *and* $0 < u \leq c$*, the u is a constant*

Proof.

$$(log(u))_{xx} = (\frac{u_x}{u})_x = \frac{uu_{xx} - u_x^2}{u^2}$$

q.e.d.

Now let us assume the metric has weakly positive curvature operator

$$R_{ijkl}\varphi_{ij}\varphi_{kl} \geq 0$$

for all $\varphi_{ij} \neq 0$,$\varphi_{ij} \in \bigwedge^2$

Remark.

1. This condition is preserved by Ricci Flow
2. In dim 3, positive curvature operator is equivalent to positive sectional curvature
3. This condition implies Harnack estimate (or in the *Kähler* case with positive bisectional curvature) we have

$$\frac{\partial}{\partial t}R + \frac{R}{t} \geq 2R_{ij}^{-1}D_iRD_jR \qquad (t > 0)$$

If we have a solution on $\alpha < t < T$, the above inequality changes to

$$\frac{\partial}{\partial t}R + \frac{R}{t-\alpha} \geq 2R_{ij}^{-1}D_iRD_jR$$

Now if we have an ancient solution, let $\alpha \to -\infty$, $t - \alpha \to \infty$, $\frac{1}{t-\alpha} \to 0$, so we have

$$\frac{\partial}{\partial t}R \geq 2R_{ij}^{-1}D_iRD_jR$$

hence R increases point-wise. So the current bound on R on a set implies bound on any early time on the same set.

Theorem 8 *If we have an ancient solution with bounded curvature and has nonnegative curvature operator, if R attains its space-time maximum at an interior point $(\bar{p}, \bar{t})$, then the solution is a steady soliton*

$$R_{ij} = D_i D_j f$$

Proof. strong maximum principle applies to Harnack estimate. q.e.d.

Next we define two important quantities for complete ancient solutions with bounded curvature and has nonnegative curvature operator.

Definition. Let us pick any point as origin, let s denote the distance to origin 0, let $B_s(0)$ denote the ball of radius s around the origin, and let $Vol(B_s(0))$ be its volume. Since the manifold has weakly positive Ricci curvature, the standard Bishop volume comparison theorem tells us that

$$\frac{Vol(B_s(0))}{s^n}$$

is monotone decreasing as s increase. We define the asymptotic Volume ratio

$$\nu = \lim_{s \to \infty} \frac{Vol(B_s(0))}{s^n}$$

Definition. Let 0, s as above, let R be the scalar curvature, we define the asymptotic scalar curvature ratio

$$A = \limsup_{s \to \infty} \; Rs^2$$

Remark. The definitions of ν and A does not depend on the choice of the origin. For more details of those two quantities, the reader can check Hamilton's survey paper [4]. We will the proof for now.

Theorem 9 *A is independent of time t under the Ricci flow. Moreover, if we assume* $|R_m| \to 0$ *as* $s \to \infty$ *(this is a condition preserved by the Ricci flow), then* ν *is also a constant under the Ricci flow.*

Proof. cf. [4]. q.e.d.

Theorem 10 *For dimension 3, if we have an ancient solution with* $|R_m| \leq c$ *and* $R_m \geq 0$*, then* $\nu = 0$ *and* $A = \infty$*.*

Proof. "blow-back" in time ($t \to -\infty$). We will divide the proof into two cases:

I) $\limsup_{t \to -\infty} |t| \sup_{M_t} R = \Omega < \infty$

II) $\limsup_{t \to -\infty} |t| \sup_{M_t} R = \infty$

(for more detail, check note P63-66.) q.e.d.

Before we prove the above theorem, we state the following theorem (cf. [4] Thm. 24.7)

Theorem 11 *Suppose we have a complete k-non-collapsed solution on some scale to the Ricci flow on a three-manifold on a maximal time interval* $0 \leq t < T < \infty$ *with bounded curvature, and*

$$(T - t)|R_m| \leq \Omega < \infty$$

then either

a) M^3 *is compact and* $M^3 \to S^3/\Gamma$ *as* $t \to T$

or

b) *M^3 (which is noncompact) has a sequence (p_j, t_j) where*

$$(T-t)|R_m(p_j,t_j)| \geq \theta > 0$$

for some θ and blow-up limit around (p_j, t_j) is $S^2 \times \mathbb{R}^1/\Gamma$.

Proof. check note P67-72. q.e.d.

Suppose we have a complete three-manifold, which is an ancient solution to the Ricci flow with bounded curvature on $-\infty < t < T$. (by the pinching estimate, we know that $R_m \geq 0$.) Let us do a blow back for $t \to -\infty$. We will have the following two cases:

I) $|R_m||t| \leq \Omega < \infty$, as $t \to -\infty$

or

II) $\limsup_{t \to -\infty} |R_m||t| = \infty$

If we take limit as $t \to -\infty$ around a sequence where scalar curvature R is maximum. We have a solution satisfies either

I) $-\infty < t < \Omega$, $|R_m| \leq \frac{\Omega}{\Omega - t}$ and $R(0,0) = 1$

or

II) $-\infty < t < \infty$, $|R_m| \leq 1$ and $R(0,0) = 1$

In case I, we can do another limit as $t \to \Omega$, this is still a backward limit of original solution and $t \to \infty$ (forward limit of a backward limit is still a backward limit) and get type I limit.

Apply previous result Thm. 11 we have either

i) converges to S^3/Γ, and the ancient solution is always S^3/Γ.

or

ii) another limit converges to $S^2 \times \mathbb{R}^1/\Gamma$, and the ancient solution is always $S^2 \times \mathbb{R}^1/\Gamma$.

Question. Is it possible that the following ancient solution (Fig. 6.1) exist? If it exists, is it Type II?

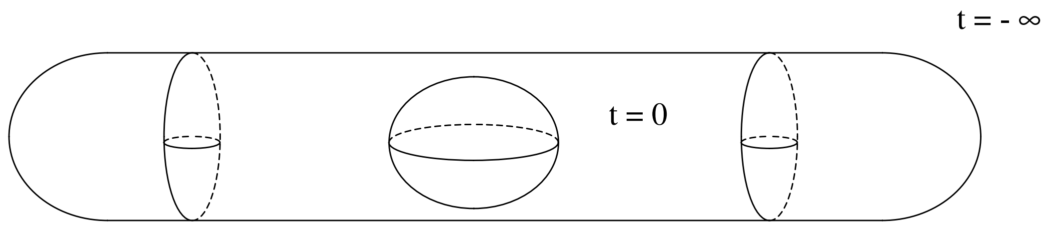

Figure 6.1: ancient solution

In case (ii), there exists backward limit rescaled around $R(p_j, t_j)$ which attains maximum at time t_j, and the limit is $S^2 \times \mathbb{R}^1/\Gamma$ (Fig. 6.2).

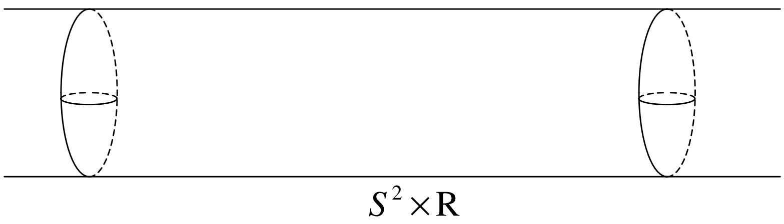

Figure 6.2: $S^2 \times \mathbb{R}$

So $\nu = 0$ and if it is k-non-collapsed, then $A = \infty$. Because if $Rs^2 \leq A < \infty$, then

$$inj\ radius \geq k \cdot \frac{s}{2\sqrt{A}} = \delta s$$

and (see Fig. 6.3)

$$Vol(B_{\frac{3s}{2}}(0)) \geq Vol(B_{\frac{s}{2}}(p)) \geq \epsilon s^n$$

hence

$$\frac{Vol(B_{\frac{3s}{2}}(0))}{(\frac{3}{2}s)^n} \geq \eta > 0$$

this is a contradiction.

In the above, we use the k-non-collapsed condition plus $\nu = 0$ and get $A = \infty$.

In case (II), since R attains its maximum at $(0,0)$, so M is a steady soliton with

$$R_{ij} = D_i D_j f \geq 0$$

By a former theorem of Hamilton (cf. eternal solution), we have

$$A = \infty$$

(otherwise the backward limit on $M^m - \{0\}$ gives a complete flat metric on $\mathbb{R}^m - \{0\}$ which we know does not exist.)

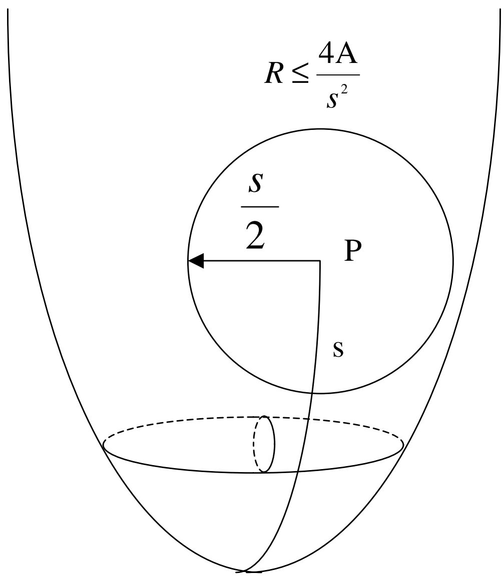

Figure 6.3: asymptotic volume ratio

Since $A = \infty$, we can do a blow-out limit as $p \to \infty$ which splits a flat factor $N^2 \times \mathbb{R}^1$, where N^2 is also an ancient solution, and Kk-non-collapsed, so it must be S^2. Now out limit is $S^2 \times \mathbb{R}^1/\Gamma$ and hence $\nu = 0$.

So in case II, we also have $A = \infty$ and $\nu = 0$.

Hence in dimension 3, we prove that $A = \infty$ and $\nu = 0$ for Kk-non-collapsed (all scales) complete ancient solutions.

2 Local Estimates for Ancient Solutions

In this section, we shall analyze complete ancient solution to the Ricci flow, which is k-non-collapsed for some k and all scales with bounded curvature on $-\infty < t \leq T < \infty$. We also assume the following conditions:

1. Nonnegative curvature operator $R_m \geq 0$
2. $\nu = 0$

Remark. In dimension 3, both conditions are satisfied automatically, for higher dimension, Perelman [5] proves the second condition is true.

Theorem 12 *(cf. [5] 11.6(a)) For all $w > 0$, $\exists C = C(w)$, such that if we have a solution satisfies all conditions described above, and for some point P and $r > 0$,*

$$Vol(B(P, r, t)) \geq wr^n \qquad \forall t \in [t_0, T]$$

then

$$R(x, t) \leq Cr^{-2} + \frac{C}{t - t_0} \qquad \forall (x, t) \in B(P, \frac{r}{2}, t)$$

Remark.

1) Here C is a universal constant, which does not depend on an particular solution of the Ricci flow

2) there is a small difference between this argument and Perelman's, which is a local estimate. (?)

3 Analysis of Ancient Solutions

(Q: Did we use $\nu = c$?

Suppose we have a solution to the Ricci flow

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$$

satisfies the following conditions:

(1) complete with bounded curvature.

(2) (ancient) the solution exists on $-\infty < t \leq T < \infty$

(3) Curvature operator $R_m \geq 0$

(4) In the class where asymptotic volume ratio

$$\nu = \lim_{s \to \infty} \frac{Vol(B_s(0))}{s^n} = 0$$

Remark.

(a) condition (3) is always true in dimension 3 by the pinching estimate.

(b) condition (4) is always true for dimension 3 (if (1)-(3) is true).

(c) Perelman proves that under k-non-collapse condition, for some k and all scales, (4) is true (if(1)-(3) is true)

(d) Hamilton conjectured that (4) is true for all n.

Exercise. If $Rs^2 < \infty$, prove by Bishop volume comparison theorem that the blow up limit will be cone-like.

Lemma 17 *For all ε, $\exists A = A(\varepsilon, k)$, such that for any solution satisfies condition (1)-(4). Inside any Ball $B(P, r)$ for some point P and radius r, if*

$$\sup_{Q \in B(P,r)} R(Q)[r - d(P,Q)]^2 \geq A$$

then

$$Vol(B(P,r)) \leq \varepsilon r^n$$

Proof. Suppose it is not true. So there exists a $\varepsilon > 0$, for any $A_j \to \infty$, we can find

$$\{M_j, g_j, P_j, r_j\}$$

such that

$$\sup_{Q \in B(P_j, r_j)} R_j(Q)[r_j - d(P_j, Q)]^2 \geq A_j$$

but

$$Vol(B(P_j, r_j)) > \varepsilon r_j^n$$

Suppose the sup attains at Q_j. i.e.,

$$R_j(Q_j)[r_j - d(P_j, Q_j)]^2 = A_j$$

and

$$R_j(Q)[r_j - d(P_j, Q)]^2 \le A_j$$

for all other Q. Rescaling such that $\widetilde{R_j(Q_j)} = 1$. Let

$$R_j(Q_j) = \frac{1}{\rho_j^2},$$

so

$$\rho_j = \frac{1}{\sqrt{R_j(Q_j)}}$$

(after rescaling, $\widetilde{\rho_j} = 1$)

Take Q_j as the new origins, and take the limit of rescaled solution. Since $\forall x$,

$$r - d(P, x) \ge (r - d(P, Q) - d(Q, x))$$

Let $d(x, Q_j) = \beta\rho_j$

$$r - d(P_j, Q_j) = \sqrt{\frac{A_j}{R_j(Q_j)}} = \sqrt{A_j} \cdot \rho_j$$

so

$$r - d(P, Q) - d(Q, x) = \sqrt{A_j}\rho_j - \beta\rho_j = (\sqrt{A_j} - \beta)\rho_j$$

Hence

$$\frac{R_j(x)}{R_j(Q_j)} \le \frac{(r_j - d(P_j, Q_j))^2}{(r_j - d(P_j, x))^2} = \frac{A_j\rho_j^2}{(\sqrt{A_j} - \beta)^2\rho_j^2} = \frac{1}{(1 - \frac{\beta}{\sqrt{A_j}})^2} \to 1$$

as $A_j \to \infty$ for fixed β. (why fixed? β is fixed means x is in finite distance from Q_j.)

Since it is k-non-collapsed, so the limit exist and $R(0) = 1$, $R \le 1$ at time T. (?)

Because of the Harnack inequality, this is also true for all $t \le T$. Hence the limit is still a solution to the Ricci flow with condition (1)-(4). i.e., $\nu = 0$ on the limit.

For given ε, there exists radium λ_0, such that, for all $\lambda \geq \lambda_0$, then

$$vol(B(0,\lambda)) \leq \epsilon\lambda^n/2^{n+1}$$

because

$$B(Q,2r) \supseteq B(p,r)$$

for r large enough. so

Claim 1

$$vol_j(B(P_j,r_j)) \leq vol(B(Q_j,2r_j)) \leq \epsilon\lambda^n/2^n$$

Since $vol_j(B(Q_j,2r_j)) \to vol(B_\infty(Q_\infty,2r))$, now take $\tilde{r_j} = \frac{r_j}{\rho_j} \to \infty$. So if j is large enough, then

$$\frac{r_j}{\rho_j} > \lambda_0$$

so

$$\frac{vol_j(B_j(Q_j,2r_j))}{(2r_j)^n} = \frac{\widetilde{vol_j}(\widetilde{B_j}(Q_j,\frac{2r_j}{\rho_j}))}{(2r_j/\rho_j)^n}$$
$$\leq \frac{\widetilde{vol_j}(\widetilde{B_j}(Q_j,\lambda_0))}{(\lambda_0)^n} \to \frac{vol_\infty(B_\infty(Q_\infty,\lambda_0))}{(\lambda_0)^n} \leq \frac{\epsilon}{2^{n+1}}$$

If j is large enough, then before convergence, we also have

$$\frac{vol_j(B_j(Q_j,2r_j))}{(2r_j)^n} \leq \frac{\epsilon}{2^n}$$

On the other hand,

$$\frac{vol_j(B_j(Q_j,2r_j))}{(2r_j)^n} \geq \frac{vol_j(B_j(P_j,2r_j))}{(2r_j)^n}$$

so $vol_j(B_j(P_j, r_j)) \leq \epsilon r_j^n$.

This is a contradiction, hence our lemma is true. q.e.d.

Theorem 13 *For all $k > 0$, all $\lambda > 0$. $\exists B = B(k, \lambda) < \infty$, such that for any solution to the Ricci flow satisfies condition (1)-(3), which is k-non-collapsed on all scales. If there exists a ball $B(x, \sigma)$ around some x and some radius σ. we have $R \leq \frac{1}{\sigma^2}$ in $B(x, \sigma)$ then $R \leq \frac{B}{\sigma^2}$ in $B(x, \lambda\sigma)$*

Remark. *In fact, $1 < \lambda < \infty$. If $\lambda \leq 1$, we can take $B = 1$.*

Proof. Because of the k-non-collapse,

$$vol(B(x, \sigma)) \leq k\sigma^n$$

Since

$$B(x, \sigma) \subseteq B(x, (\lambda + 1)\sigma)$$

choose

$$\epsilon < \frac{k}{(\lambda + 1)^n}$$

by the previous lemma: $\exists A = A(\epsilon, k)$. Since for Ball $B(x, (\lambda + 1)\sigma)$,

$$vol(B(x, (\lambda + 1)\sigma)) \leq k\sigma^n > \epsilon((\lambda + 1)\sigma)^n$$

so

$$R(Q)[(\lambda + 1)\sigma - d(x, Q)]^2 \leq A$$

for any $Q \in B(x, (\lambda + 1)\sigma)$, so if $Q \in B(x, \lambda\sigma)$, then $d(x, Q) \leq \lambda\sigma$. so

$$(\lambda + 1)\sigma - d(x, \sigma) \geq \sigma$$

so

$$R(Q)\cdot\sigma^2 \le R(Q)[(\lambda+1)\sigma - d(x,\sigma)]^2 \le A$$

hence

$$R(Q) \le \frac{A}{\sigma^2}$$

we only need to take $B = A$. q.e.d.

Theorem 14 *For all $k > 0$, $\exists a = a(k) > 0$, such that if we have a solution to the Ricci flow satisfies conditions (1)-(3) and is K-non-collapse on all scales. Then for any point x, let $r = r(x)$ be the largest radius r such that $R \le \frac{1}{r^2}$ in $B(x,r)$. We also have*

$$R(x) \ge \frac{a}{r^2}$$

Remark. If we let $\sigma = \sqrt{a}/\sqrt{R(x)}$, then $r \ge \sqrt{a}/\sqrt{R(x)}$. Since $R \le \frac{1}{r^2}$ in $B_r(x)$ and $r \ge \sqrt{a}/\sqrt{R(x)}$, we have $\frac{1}{r^2} \le \frac{1}{a}R(x)$, hence $R(Q) \le \frac{1}{a}R(x)$, $\forall Q \in B(x, \sqrt{\frac{a}{R(x)}}$, so we have the following corollary:

Corollary 12 *$\forall k > 0$, $\exists 0 < C = C(k) < \infty$, such that $\forall Q \in B(x, \frac{c}{R(x)})$, we have $R(Q) \le CR(x)$.*

Proof. Since r is the largest radius, such that $R \le \frac{1}{r^2}$ in $B(x,r)$, so there exists a point Z in $B(x,r)$ such that $R(Z) = \frac{1}{r^2}$, otherwise r could be larger.

Now by the last theorem, take $\lambda = 4$, there exists a constant $A = A(k,4) < \infty$, such that $R \le \frac{A}{r^2}$ in $B(x,4r)$. Notice because $d(x,Z) \le r$, so $B(Z,3r) \subset B(x,4r)$. The aboves are true at time T, now we need to think a little bit backward. Since we have positive curvature, keep in mind that the distance is shrinking.

We can find $\delta > 0$ small enough, such that for $T - \delta r^2 \le t \le T$, $B(Z,2r,t) \subseteq B(Z,2r,T)$. On the other hand, we can also find $\delta > 0$ small enough, such that if $T - \delta r^2 \le t \le T$, then $d(Z,x,t) \le 2r$.

In fact, we know the first one is always true, so we only need to choose δ satisfies the second one.

Now we use $W.X.Shi's$ derivative estimate, which gives bound on $|DR|$, $|D^2R|$ and $|D_tR|$ in $B(Z, 2r, T - \frac{1}{2}\delta r^2 \leq t \leq T)$, so we have

$$R(Z,t) \geq \frac{1}{2r^2}, \forall T - \frac{\delta}{2}r^2 \leq t \leq T$$

(In fact, $|DR| \leq C_1/r^3$, $|D^2R|$,$|D_tR| \leq \frac{C_2}{r^4}$).

Let's pick a geodesic from X to Z at time $(T - \frac{1}{2}\delta r^2)$, this is the largest geodesic between X and Z for all $t \in [T - \frac{\delta}{2}r^2, T]$. At time $t = T$, the length of this geodesic is less than r.

We can parametrize it by the arc length s at time T. So

$$Y(s) : Y(0) = Z,\ Y(\tilde{s}) = X.$$

$|\frac{dY}{dS}|_{t=T} = 1$, $0 \leq s \leq \bar{s} \leq d(z, x, T) \leq r$.

Now we take the path in space-time:

$$[Y(s), T - \frac{1}{2}\delta r^2 + \frac{s}{\bar{s}}(\frac{1}{2}\delta r^2)]$$

We have $|\frac{dY}{dS}|_t \leq 2$ for all $t \in [T - \frac{1}{2}\delta r^2, T]$.

This is because we have bounded curvature, so length does not change too much, we can make δ smaller if needed. Since the curvature operator is positive, we have Harnack inequality:

$$\frac{\partial R}{\partial t} \geq 2R_c^{-1}(DR, DR) \geq \frac{2}{R}|DR|^2 \geq \frac{2r^2}{A}|DR|^2$$

($R_c \geq 0$, $R_c \leq R_g$, $R_c^{-1} \geq \frac{1}{R} g^{-1}$ and $R \leq A/r^2$)

This is the true for any (p,t) in $B(x,4r) \times [T - \frac{1}{2}\delta r^2, T]$. Now along the space-time path:

$$\begin{aligned}\frac{dR}{dt} =& \frac{\partial R}{\partial t} + DR(Y)\frac{dY}{ds} \cdot \frac{ds}{dt} \\ \geq& \frac{\partial r^2}{A}|DR(Y)|^2 - |DR(Y)| \cdot 2 \cdot \frac{2r}{\delta r^2} \\ \geq& -\frac{1}{r^4} \cdot \frac{A}{2} \cdot \frac{1}{\delta^2}\end{aligned}$$

Here we use

$$\begin{aligned} t &= T - \frac{1}{2}\delta r^2 + \frac{s}{\bar{s}}(\frac{1}{2}\delta r^2) \\ \frac{dt}{ds} &= \frac{1}{2}\delta r^2 / \bar{s} = \frac{\delta r^2}{2\bar{s}} \\ \frac{ds}{dt} &= \frac{s\bar{s}}{\delta r^2} \leq \frac{2r}{\delta r^2}\end{aligned}$$

Or we can write

$$\frac{dR}{dt} \geq \frac{2}{R}|DR|^2 - |DR| \cdot 2 \cdot \frac{2}{\delta r} \geq -\frac{2}{\delta^2} \cdot \frac{1}{r^2} R$$

i.e. $\frac{d}{dt} lnR \geq -\frac{2}{\delta^2}\frac{1}{r^2}$.

$$lnR(x,t) - lnR(Z, T - \frac{1}{2}\delta r^2) \geq -\frac{2}{\delta^2}\frac{1}{r^2}(\frac{1}{2}\delta r^2) = -\frac{1}{\delta}$$

So $R(x,T) \geq R(Z, T - \frac{1}{2}\delta r^2) \cdot e^{-\frac{1}{\delta}} \geq \frac{1}{2r^2} e^{-\frac{1}{\delta}}$.

Take $a = \frac{1}{2} e^{-\frac{1}{\delta}}$, this finishes the proof of the theorem.

q.e.d.

Remark. Without the assumption of K-non-collapse, this theorem is not true. For example, look at cigar.

Suppose it is true, since $R(x) = e^{-s}$ (assume $M = 1$), $\sqrt{R(x)} = e^{-s/2}$, so $r = \sqrt{a}R(x) = \sqrt{a}e^{-s}$.

Now at $s - r = s - \frac{\sqrt{a}}{e^{-s/2}}$.

$$R(Y) = e^{-[s-r]} = e^{-[s-\frac{\sqrt{a}}{e^{-s/2}}]} = e^{-s} \cdot e^{\sqrt{a}e^{s/2}} = R(x)e^{\sqrt{a}e^{s/2}}$$

If there exists such a, then $\frac{R(Y)}{R(x)} \to \infty$ as $s \to \infty$. One reason is that $DR = \frac{\partial R}{\partial s} = e^{-s}$, so $\frac{|DR|^2}{R^3} = e^s \to \infty$ as $s \to \infty$

Bibliography

[1] Dennis M. DeTurck. Deforming metrics in the direction of their Ricci tensors. *J. Differential Geom.*, 18(1):157–162, 1983.

[2] Richard S. Hamilton. Three-manifolds with positive Ricci curvature. *J. Differential Geom.*, 17(2):255–306, 1982.

[3] Richard S. Hamilton. Four-manifolds with positive curvature operator. *J. Differential Geom.*, 24(2):153–179, 1986.

[4] Richard S. Hamilton. The formation of singularities in the Ricci flow. In *Surveys in differential geometry, Vol. II (Cambridge, MA, 1993)*, pages 7–136. Internat. Press, Cambridge, MA, 1995.

[5] Grisha Perelman. The entropy formula for the Ricci flow and its geometric applications. *Preprint*, 2002.